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Higher derivatives and brane-localised kinetic terms in gauge theories on orbifolds

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Abstract: We perform a detailed analysis of one-loop corrections to the self-energy of the (off-shell) gauge bosons in six-dimensional $\mathcal{N}=1$ supersymmetric gauge theories on orbifolds. After discussing the Abelian case in the standard Feynman diagram approach, we extend the analysis to the non-Abelian case by employing the method of an orbifold-compatible one-loop effective action for a classical background gauge field. We find that bulk higher derivative and brane-localised gauge kinetic terms are required to cancel one-loop divergences of the gauge boson self energy. After their renormalisation we study the momentum dependence of both the higher derivative coupling $h(k^2)$ and the *effective* gauge coupling $g_{\rm eff}(k^2)$. For momenta smaller than the compactification scales, we obtain the 4D logarithmic running of $g_{\rm eff}(k^2)$, with suppressed power-like corrections, while the higher derivative coupling is constant. We present in detail the threshold corrections to the low energy gauge coupling, due to the massive bulk modes. At momentum scales above the compactification scales, the higher derivative operator becomes important and leads to a power-like running of $g_{\rm eff}(k^2)$ with respect to the momentum scale. The coefficient of this running is at all scales equal to the renormalised coupling of the higher derivative operator which ensures the quantum consistency of the model. We discuss the relation to the similar one-loop correction in the heterotic string, to show that the higher derivative operators are relevant in that case too, since the field theory limit of the one-loop string correction does not commute with the infrared regularisation of the (on-shell) string result.

Keywords: Field Theories in Higher Dimensions, Beyond Standard Model, Supersymmetric gauge theory, Large Extra Dimensions.

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Contents

1. Introduction

In recent years, the study of additional compact space dimensions in an effective field theory framework [1] has been popular in the particle physics community, since this provides new possibilities for searching for physics beyond the Standard Model. Although string theory may present a better set-up for such studies, effective field theories also allow a fully consistent investigation of quantum effects associated with (large) extra dimensions, and may even capture effects not seen by the *on-shell* string. Since no additional space dimensions are observed at low energies, these have to be compactified at sufficiently high scales[1]. In field theory approaches only simple covering spaces are usually considered, such as S^1, $T^2\ldots$, sufficient however to capture the main effects investigated. To obtain 4D chiral fermions from bulk fields discrete symmetries must act (non-freely) upon the extra dimensions, resulting in orbifolds such as $S^1/\mathbb{Z}_2$ or $T^2/\mathbb{Z}_N$ $(N = 2, 3, 4, 6)$. These orbifolds have fixed points, invariant under subgroups of the discrete group action. Since the bulk fields satisfy boundary conditions at the orbifold fixed points, momentum conservation does not hold in the extra dimensions. Ultimately, brane-localised (either 4D or higher derivative) interactions are required as counterterms [3 − 7], to ensure the quantum consistency of the models. New bulk interactions, in addition to the original ones, are also generated dynamically [7 − 12] as higher dimensional (derivative) terms.

In this paper we consider the one-loop correction to the self-energy of gauge bosons in 6D $\mathcal{N} = 1$ supersymmetric Abelian and non-Abelian gauge theories coupled to hypermultiplets on the $T^2/\mathbb{Z}_2$ orbifold, within the component field formulation. We find that one-loop divergences are generated which require the addition of new counterterms. These involve new, brane-localised 4D interactions as well as higher derivative, bulk gauge interactions, not present in the original action. We provide a careful study of the role of these operators in the running of the gauge coupling at high and low momentum scales. We also discuss the link between these one-loop corrections and those in string theory. These are the main purposes of this paper. Recent work on this topic can be found in [9, 10] in the superfield formalism (for related studies see also [13]).

In the Abelian case, we use the Feynman diagram approach to consider bulk scalar and fermion contributions to the self-energy of the gauge bosons. We find that the fermions give rise to a bulk divergence only, requiring a bulk higher derivative counterterm. At the technical level, the origin of this divergence is the presence of infinite double sums over the modes and a re-summation of their individual divergent contributions [5 − 7, 9 − 11, 14]. In contrast, bulk complex scalars bring in both bulk and brane corrections. Their divergent part must be cancelled by bulk higher derivative and brane-localised gauge kinetic counterterms, respectively. Both fermionic and bosonic contributions also contain finite Lorentz violating mass terms in the bulk. For a hypermultiplet there are neither brane contributions nor bulk Lorentz violating mass terms. Thus, even after compactification, the Lorentz invariance in these mass corrections is protected by the initial supersymmetry. Nonetheless, one still needs a bulk higher derivative counterterm, which reflects the non-renormalisable nature of the initial, higher dimensional field theory.

[1] Non-compact, infinite extra dimensions are also possible [2].

The above analysis is extended to the non-Abelian case by employing a background field method which is made consistent with the orbifold boundary conditions. This formalism can be generalised to other orbifold actions, such as Wilson lines. The results show that a hypermultiplet generates only a bulk loop correction, just like in the Abelian case, while a vector multiplet generates both bulk and brane-localised contributions. These contributions contain divergent terms which are cancelled by bulk higher derivative and brane-localised gauge kinetic counterterms. After the renormalisation of these operators, the running of the one-loop *effective* coupling $g_{\text{eff}}(k^2)$, which is the coupling of the zero mode gauge bosons, is controlled by finite terms coming from both bulk and branes. This will be discussed in detail.

In the limit of external momenta k^2 smaller than the compactification scale(s), the higher derivative gauge kinetic term is suppressed. In this case, after considering both bulk and brane one-loop effects, we show that the *effective* gauge coupling has a 4D *logarithmic* running with respect to the momentum k^2, with the 4D $\mathcal{N}=1$ beta function. This is an interesting result and a consistency check of our calculation. The logarithmic running in momentum originates from both bulk and brane contributions. We also establish a relation between the high scale physics (g_{tree}) and $g_{\text{eff}}(k^2 \ll 1/R_{5,6}^2)$, which involves re-summing threshold corrections due to infinitely many massive Kaluza-Klein modes. We provide detailed expressions of these corrections including finite terms. This relation is little dependent on the role of the higher derivative operator, strongly suppressed at such low momentum scales. The running of the effective coupling with respect to k^2 can be extended to larger values of k^2, closer to compactification scales ($k^2 \sim 1/R_{5,6}^2$), to reach the regime of dimensional cross-over [15]. In this case the higher derivative operator brings in an important contribution to the effective gauge coupling. After its renormalisation, there are non-negligible power-like corrections in momentum scale to $g_{\text{eff}}(k^2)$. The coefficient of the power-like running is the renormalised coupling $h(k^2)$ of the higher derivative operator, which below the compactification scales is constant while far above them it runs logarithmically with respect to the momentum scale. At even higher momentum scales $k^2 \gg 1/R_{5,6}^2$ we show that $g_{\text{eff}}(k^2)$ has a power-like running with respect to the high momentum scale, with a coefficient equal to the renormalised coupling of the higher derivative operator.

The link of these corrections to similar results from string theory is addressed. We discuss the relation of our result to string corrections in the type I strings [16] and heterotic toroidal orbifolds [17, 18] with $\mathcal{N}=2$ sub-sectors. Although the *on-shell* (heterotic) string calculation to the gauge boson self-energy misses contributions associated with higher derivative operators, we show that there are remnant effects of their presence, even in the (on-shell) string result. These effects are related to the fact that the infrared regularisation of the (heterotic) string loop corrections and their $\alpha' \to 0$ limit do not commute, leaving a troublesome UV-IR mixing in the effective field theory regime of the (heterotic) string ($\alpha' \to 0$). This stresses the importance of investigating the role of such operators in string theory, too.

The results for the self-energy of the gauge bosons in our component field formulation are fully consistent with those obtained in the superfield formulation. Nevertheless, the gauge fixing term and the associated ghost Lagrangian considered are not invariant under the original supersymmetry transformation. This is related to the well-known fact that the Wess-Zumino gauge is not consistent with a super-covariant gauge fixing [19]. This problem is very common in similar works, and becomes manifest in the fact that the anomalous dimensions of scalar and fermion matter fields in a chiral multiplet are not equal at one-loop level [20]. However, for our case of the self-energy of the gauge bosons, additional auxiliary multiplets required by a manifestly supersymmetric quantisation will not change the result, as discussed for the holomorphic anomaly to the gauge coupling in 4D supersymmetric gauge theory [21].

The paper is organised as follows. We start with a 6D $\mathcal{N}=1$ supersymmetric Abelian gauge theory where the one-loop correction to the gauge bosons is computed. Then we employ the higher dimensional background field method to find the one-loop effective action of non-Abelian gauge theories and apply this formalism to $T^2/\mathbb{Z}_2$, using orbifold-compatible functional differentiations. Finally we discuss the running of the effective gauge coupling. Technical details of our calculations are given in the appendices.

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2. One-loop vacuum polarisation to U(1) gauge bosons on orbifolds

We consider the one-loop vacuum polarisation in a 6D $\mathcal{N}=1$ supersymmetric Abelian gauge theory coupled to hypermultiplets. The two extra dimensions are denoted by the complex coordinate $z=x_5+ix_6$, and are compactified on the orbifold $T^2/\mathbb{Z}_2$ with the two radii R_5 and R_6. The torus is modded out by the $\mathbb{Z}_2$ reflection, which identifies coordinates of extra dimensions under $z\to -z$. Under this $\mathbb{Z}_2$ action, there appear four fixed points which transform into themselves.

In a 6D $\mathcal{N}=1$ supersymmetric gauge theory, a vector multiplet is composed of gauge bosons A_M and (right-handed) symplectic Majorana gauginos λ while a hypermultiplet is composed of two complex hyperscalars $\phi_\pm$ with opposite charges and a (left-handed) hyperino ψ. The supersymmetric action is given in component fields[2] by [22]

$$S = S_{\text{vector}} + S_{\text{hyper}}$$

with

$$S_{\text{vector}} = \frac{1}{2}\int d^6X\Big[-\frac{1}{2}F_{MN}F^{MN}+\bar{\lambda}i\gamma^M\partial_M\lambda+\bar{\lambda}^c i\gamma^M\partial_M\lambda^c+\left|D^1+iD^2\right|^2+(D^3)^2\Big], \tag{2.1}$$

$$\begin{aligned} S_{\text{hyper}} = \int d^6X\Big[&\sum_{\pm}|D_M\phi_\pm|^2+\bar{\psi}i\bar{\gamma}^M D_M\psi+\sqrt{2}g\Big(\bar{\psi}\lambda\phi_-^*+\bar{\psi}\lambda^c\phi_+ +\text{c.c.}\Big)\\ &-g\Big((D^1+iD^2)\phi_+\phi_- +\text{c.c}\Big)+gD^3\Big(\phi_+^*\phi_+-\phi_-^*\phi_-\Big)\Big], \end{aligned} \tag{2.2}$$

where $\lambda^c = C_5\bar{\lambda}^T$ is the five-dimensional charge conjugate of λ, $D_M\phi_\pm = (\partial_M \mp igA_M)\phi_\pm$, and $D_M\psi = (\partial_M - igA_M)\psi$. Details on our conventions are given in appendix A.

[2] We also included the auxiliary fields $\vec{D}=(D^1,D^2,D^3)$ for completeness. We have written gaugino and hyperino in 4D Dirac representations.

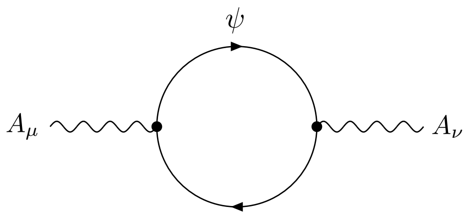

Figure 1: The Feynman diagram with a bulk fermion ψ contributing to $\Pi_{\mu\nu}$ at one-loop order.

To promote the $\mathbb{Z}_2$-symmetry of the orbifold to a symmetry of our theory, we have to specify the $\mathbb{Z}_2$ parities of the bulk fields. These parities are given by

$$\begin{gathered} A_\mu(x,-z) = A_\mu(x,z), \quad A_{5,6}(x,-z) = -A_{5,6}(x,z), \quad \lambda(x,-z) = i\gamma_5\lambda(x,z), \\ \phi_\pm(x,-z) = \pm\eta\,\phi_\pm(x,z), \quad \psi(x,-z) = i\eta\,\gamma_5\psi(x,z) \end{gathered} \tag{2.3}$$

where η can be chosen $+1$ or -1. Within this framework, we evaluate the contributions to the 4D one-loop self-energy of the gauge bosons induced by bulk fields running in the loop.

2.1 A bulk fermion contribution

We consider the one-loop contribution of a 6D left-handed bulk fermion to the self-energy of the 4D components of the gauge field. The Feynman diagram given in figure 1 can be evaluated as

$$\begin{aligned} \Pi^f_{\mu\nu}(k,\vec{k},\vec{k}') = g^2\mu^{4-d}\sum_{\vec{p},\vec{p}'}\int\frac{d^dp}{(2\pi)^d}\,\mathrm{Tr}\Big\{\gamma_\mu\frac{i}{2}\Big[\frac{\delta_{\vec{p},\vec{p}'}}{\not{p}+\gamma_5p_5+p_6} - \eta\frac{\delta_{\vec{p},-\vec{p}'}}{\not{p}+\gamma_5p_5+p_6}i\gamma_5\Big]\gamma_\nu \\ \times\frac{i}{2}\Big[\frac{\delta_{\vec{k}'+\vec{p}',\vec{k}+\vec{p}}}{\not{p}+\not{k}+\gamma_5(k_5'+p_5')+k_6'+p_6'} - \eta\frac{\delta_{\vec{k}'+\vec{p}',-\vec{k}-\vec{p}}}{\not{p}+\not{k}+\gamma_5(k_5'+p_5')+k_6'+p_6'}i\gamma_5\Big]\Big\} \end{aligned} \tag{2.4}$$

where we used eq. (B.4) for the fermion propagator in the loop. Here a sum over discrete momenta $\vec{p}$ is to be understood as a double sum over integers $n_{1,2}$ such that for an arbitrary function f

$$\sum_{\vec{p}} f(\vec{p}) = \sigma\sum_{n_{1,2}\in\mathbf{Z}} f(n_1/R_1, n_2/R_6), \qquad \sigma \equiv [(2\pi)^2R_5R_6]^{-1} \tag{2.5}$$

where $\vec{p} \equiv (p_5,p_6) = (n_1/R_5, n_2/R_6)$. Moreover, we use the Kronecker delta symbol for discrete momenta, whose action and normalisation are

$$\sum_{\vec{p}}\delta_{\vec{p},\vec{p}'}f(\vec{p}) = f(\vec{p}'), \qquad \delta_{\vec{p},\vec{p}'} \equiv (2\pi)^2\delta_{p_5,p_5'}\delta_{p_6,p_6'} = \frac{1}{\sigma}\,\delta_{n_1,n_1'}\delta_{n_2,n_2'} \tag{2.6}$$

The integral in (2.4) is continued to $d \equiv 4-\epsilon$ dimensions, with $\epsilon \to 0$ *after* performing the double sum; μ is the finite scale of the DR scheme. Note that *both* the 4D integral *and* the double sum over the momenta are regularised by the same regulator ϵ. That is, ϵ acts essentially as a 6D regulator, as it should be the case. These conventions will be used throughout the paper. After some standard calculations, we rewrite expression (2.4) as

$$\Pi^f_{\mu\nu} = -\frac{1}{4}g^2 \sum_{\vec{p},\vec{p}'} \int \frac{d^d p}{(2\pi)^d} \frac{\mu^{4-d}}{(p^2-p_5^2)[(p+k)^2-(\vec{p}'+\vec{k}')^2]} \Big\{ \pi^{(1)}_{\mu\nu}(\vec{p}',\vec{k}')\delta_{\vec{k}',\vec{k}}$$
$$+\pi^{(1)}_{\mu\nu}(-\vec{p}',-\vec{k}')\delta_{\vec{k}',-\vec{k}} - \eta\pi^{(2)}_{\mu\nu}(\vec{p}',\vec{k}')\delta_{-2\vec{p}',\vec{k}'-\vec{k}} - \eta\pi^{(2)}_{\mu\nu}(-\vec{p}',-\vec{k}')\delta_{-2\vec{p}',\vec{k}'+\vec{k}} \Big\} \tag{2.7}$$

with

$$\pi^{(1)}_{\mu\nu}(\vec{p}',\vec{k}') = 4[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu + g_{\mu\nu}(-p(p+k)+\vec{p}'\cdot(\vec{p}'+\vec{k}'))],$$
$$\pi^{(2)}_{\mu\nu}(\vec{p}',\vec{k}') = -4ip^\rho k^\sigma \epsilon_{\mu\rho\nu\sigma}. \tag{2.8}$$

Here we note that terms proportional to $\delta_{\vec{k},\vec{k}'}$ or $\delta_{\vec{k},-\vec{k}'}$ conserve the external extra momentum $|\vec{k}|$. Therefore these terms correspond to bulk terms. On the contrary, terms multiplied by $\delta_{-2\vec{p}',\vec{k}'-\vec{k}}$ or $\delta_{-2\vec{p}',\vec{k}'+\vec{k}}$ change the external discrete momentum in the compact dimensions, and therefore correspond to brane-localised terms [3]. These momentum non-conserving terms are due to the breaking of translational invariance along the extra dimensions in the presence of orbifold fixed points. Although the momentum is conserved at each vertex in Feynman diagrams, extra momenta of ingoing and outgoing gauge bosons can be different due to the momentum non-conserving part $\delta_{\vec{p},-\vec{p}'}$ in the propagator of a bulk field running in loops.

After performing the 4D momentum integral, the contribution involving $\pi^{(2)}_{\mu\nu}$ vanishes. Therefore no correction to the localised gauge coupling is generated by the bulk fermion. Finally, after introducing a Feynman parameter and shifting the integration momentum as in appendix C.1, we obtain the correction

$$\Pi^f_{\mu\nu}[k,\vec{k},\vec{k}'] = -2\,g^2\,\delta_{\vec{k},\vec{k}'}\,\mu^{4-d}\sum_{\vec{p}'}\int_0^1 dx \int \frac{d^d p}{(2\pi)^d}\frac{1}{(p^2-\Delta)^2}$$
$$\times\Big[2x(1-x)[(k^2-\vec{k}'^2)g_{\mu\nu}-k_\mu k_\nu]+(1-2x)\vec{k}'\cdot(\vec{p}'+x\vec{k}')g_{\mu\nu}\Big] \tag{2.9}$$

with $\Delta \equiv -x(1-x)(k^2-\vec{k}'^2)+(\vec{p}'+x\vec{k}')^2$. The first part of this result contains the familiar tensor structure coming from 6D gauge and Lorentz invariance and can be factorised out of the momentum integration and the Kaluza-Klein summation. The second part of (2.9) however corresponds to a Lorentz violating mass term, since 6D Lorentz invariance is broken by the compactification. This term leads to radiative corrections to the nonzero Kaluza-Klein masses [4].

The current form of the result in eq. (2.9) is all we need for our purpose of investigating the one-loop corrections to gauge couplings in supersymmetric models. It is nevertheless important to simplify eq. (2.9) to identify its divergences[3]. After some algebra we find, in

[3]The non-zero external momenta $(k,\vec{k},\vec{k}')$ in the Green functions ensure infrared-convergent integrals.

Euclidean space[4]

$$\Pi^f_{\mu\nu}[k,\vec{k},\vec{k}'] = -\frac{2g^2 i\pi^2}{(2\pi)^d}\,\sigma\,\delta_{\vec{k},\vec{k}'}\,\Big[[(k^2+\vec{k}'^2)\delta_{\mu\nu}+k_\mu k_\nu]\;\Pi^f_0-\delta_{\mu\nu}\,\Pi^f_1\Big], \tag{2.10}$$

$$\Pi^f_0 \equiv \int_0^1 dx\;\rho_0(x)\,\mathcal{J}_0[x(1-x)(k^2+\vec{k}'^2); xk'_5R_5, xk'_6R_6], \tag{2.11}$$

$$\Pi^f_1 \equiv \frac{k'_5}{R_5}\int_0^1 dx\;\rho_1(x)\,\mathcal{J}_1[x(1-x)(k^2+\vec{k}'^2); xk'_5R_5, xk'_6R_6] + \Big(k'_5\leftrightarrow k'_6; R_5\leftrightarrow R_6\Big), \tag{2.12}$$

with $\rho_0(x)\equiv 2x(1-x)$ and $\rho_1(x)\equiv(1-2x)$. The functions $\mathcal{J}_{0,1}[c;c_1,c_2]$ are defined and studied in detail in appendix D, eqs. (D.1), (D.20) to (D.24) and they can be integrated over x, yielding compact final expressions. Since these expressions are rather long, we do not present them here. However, it is important for our purpose to notice that $\mathcal{J}_0$ has a pole, while $\mathcal{J}_1$ is actually finite. Using this information, the pole structure in ϵ of the final result is obtained

$$\Pi^f_0 = \frac{\pi}{15}(k^2+\vec{k}'^2)R_5R_6\left[\frac{-2}{\epsilon}\right]+\mathcal{O}(\epsilon^0), \qquad \Pi^f_1=\mathcal{O}(\epsilon^0) \tag{2.13}$$

with momentum again in Euclidean space. The consequence of this 6D divergence in Π^f_0 and thus in $\Pi^f_{\mu\nu}$ is that a higher derivative counterterm is necessary. This is a dimension-six bulk counterterm, and its structure would be, in a non-susy case, $R_5R_6F^{MN}\Box_6F_{MN}$. Although each bulk mode brings a pole for the usual gauge kinetic term, the resummation of infinitely many bulk mode contributions leads only to a pole for the higher derivative term[5]. A similar result has been obtained in a 6D Abelian gauge theory without compactification in [8]. We postpone a further discussion on such operators to sections 2.3 and 3 where their role will be investigated in detail.

2.2 A bulk scalar contribution

Now we consider the one-loop contribution of a complex bulk scalar with parity η to the self-energy of the gauge boson. In this case, there are two Feynman diagrams (see figure 2) contributing to the one-loop vacuum polarisation.

Then the one-loop scalar contribution is

$$\Pi^s_{\mu\nu,\pm}[k,\vec{k},\vec{k}'] = \Pi^{(1)}_{\mu\nu}[k,\vec{k},\vec{k}'] + \Pi^{(2)}_{\mu\nu}[k,\vec{k},\vec{k}'] \tag{2.14}$$

[4]Denoting by Δ_E the Euclidean form of Δ we used that: $\int d^dp\,(p^2-\Delta)^{-2} = i\pi^2\int_0^\infty dt\,t^{1-d/2}\,e^{-\pi\,t\,\Delta_E}$. Unless stated otherwise, our formulae are always written using the Minkowskian metric; the distinction is also obvious by the presence of either $g_{\mu\nu}$ or $\delta_{\mu\nu}$.

[5]As will be discussed in detail in section 5, in a regularisation scheme with a momentum cutoff, note that there is no logarithmically divergent correction to the $F^{MN}F_{MN}$ operator and this is consistent with the absence of a $1/\epsilon$ pole to this operator in DR. In such cutoff regularisation, however, there exists a quadratically divergent correction to the $F^{MN}F_{MN}$ operator (unlike in the 4D gauge theory), discussed in section 5.

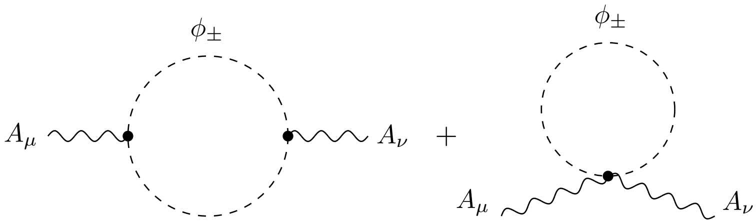

Figure 2: The Feynman diagrams with the bulk scalar ϕ contributing to $\Pi_{\mu\nu}$ at one-loop order.

with

$$\Pi^{(1)}_{\mu\nu}[k,\vec{k},\vec{k}'] = (-ig)^2\,\mu^{4-d}\sum_{\vec{p},\vec{p}'}\int\frac{d^dp}{(2\pi)^d}(2p+k)_\mu(2p+k)_\nu\frac{i}{2}\left[\frac{\delta_{\vec{p},\vec{p}'}\pm\eta\delta_{\vec{p},-\vec{p}'}}{p^2-\vec{p}^2}\right]$$
$$\times\frac{i}{2}\left[\frac{\delta_{\vec{p}'+\vec{k}',\vec{p}+\vec{k}}\pm\eta\delta_{\vec{p}'+\vec{k}',-\vec{p}-\vec{k}}}{(p+k)^2-(\vec{p}'+\vec{k}')^2}\right], \tag{2.15}$$

$$\Pi^{(2)}_{\mu\nu}[k,\vec{k},\vec{k}'] = (2ig^2)\,g_{\mu\nu}\,\mu^{4-d}\sum_{\vec{p},\vec{p}'=\vec{p}+\vec{k}-\vec{k}'}\int\frac{d^dp}{(2\pi)^d}\frac{i}{2}\left[\frac{\delta_{\vec{p},\vec{p}'}\pm\eta\delta_{\vec{p},-\vec{p}'}}{p^2-\vec{p}^2}\right] \tag{2.16}$$

where we used eq. (B.7) for the scalar propagator in the loop. After re-arranging the result, we obtain the one-loop vacuum polarisation as

$$\Pi^s_{\mu\nu,\pm}[k,\vec{k},\vec{k}'] = -\frac{g^2}{2}\mu^{4-d}\sum_{\vec{p}'}\int\frac{d^dp}{(2\pi)^d}\frac{\delta_{\vec{k},\vec{k}'}\pm\eta\delta_{-2\vec{p}',\vec{k}'-\vec{k}}}{(p^2-(\vec{p}')^2)[(p+k)^2-(\vec{p}'+\vec{k}')^2]}$$
$$\times\Big\{-(2p+k)_\mu(2p+k)_\nu+2g_{\mu\nu}\Big[(p+k)^2-(\vec{p}'+\vec{k}')^2\Big]\Big\}$$
$$\equiv \Pi^{\rm bulk}_{\mu\nu}[k,\vec{k},\vec{k}']\pm\eta\Pi^{\rm brane}_{\mu\nu}[k,\vec{k},\vec{k}'] \tag{2.17}$$

with the bulk and brane terms easily identified by whether they do or do not conserve the discrete momenta associated with the two compact dimensions. After using a Feynman parameter and a shift of the integration momentum we obtain the bulk correction, where a 6D Lorentz violating mass term is present again, due to compactification:

$$\Pi^{\rm bulk}_{\mu\nu}[k,\vec{k},\vec{k}'] = -\frac{g^2}{2}\delta_{\vec{k},\vec{k}'}\,\mu^{4-d}\sum_{\vec{p}'}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-\Delta)^2}$$
$$\times\Big[(1-2x)^2[(k^2-\vec{k}'^2)g_{\mu\nu}-k_\mu k_\nu]+2(2x-1)\vec{k}'\cdot(\vec{p}'+x\vec{k}')g_{\mu\nu}\Big]. \tag{2.18}$$

As in the fermionic case, the form of the result in (2.18) is all we need for our purpose of investigating one-loop corrections to the gauge couplings in supersymmetric models. This result can however be evaluated explicitly as done in the fermionic case, to identify its divergences and finite parts[6]. One finds, using an Euclidean metric

$$\Pi^{\rm bulk}_{\mu\nu}[k,\vec{k},\vec{k}'] = -\frac{g^2}{2}\frac{i\pi^2}{(2\pi)^d}\,\sigma\,\delta_{\vec{k},\vec{k}'}\left[[(k^2+\vec{k}'^2)\delta_{\mu\nu}+k_\mu k_\nu]\;\Pi^{\rm bulk}_0 - \delta_{\mu\nu}\,\Pi^{\rm bulk}_1\right]$$

$$\Pi^{\rm bulk}_0 = \frac{\pi}{30}(k^2+\vec{k}'^2)R_5R_6\left[\frac{-2}{\epsilon}\right]+\mathcal{O}(\epsilon^0), \qquad \Pi^{\rm bulk}_1 = \mathcal{O}(\epsilon^0) \tag{2.19}$$

Here $\Pi^{\rm bulk}_0$ and $\Pi^{\rm bulk}_1$ have an expression identical to that of Π^f_0 of (2.11) and Π^f_1 of (2.12) respectively, but with $\rho_0(x)=(1-2x)^2$, $\rho_1(x)=2(2x-1)$. The divergence of $\Pi^{\rm bulk}_{\mu\nu}$ requires a higher derivative counterterm, of structure identical to that for fermions: $R_5R_6F^{MN}\Box_6F_{MN}$. We return to discuss the role of such operators in sections 2.3), (3.

For the brane correction the Kaluza-Klein loop momentum $\vec{p}'$ is fixed by the difference between ingoing and outgoing Kaluza-Klein momenta $\vec{k}$ and $\vec{k}'$. After introducing a Feynman parameter and shifting the 4D momentum, we also find the brane correction as

$$\begin{aligned}\Pi^{\rm brane}_{\mu\nu}[k,\vec{k},\vec{k}'] =& -\frac{g^2}{2}\mu^{4-d}\sum_{\vec{p}'}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-\Delta)^2}\Big[2(1-3x+2x^2)(k^2-\vec{k}'^2)g_{\mu\nu}\\ &-(1-2x)^2k_\mu k_\nu+4(x-1)\vec{k}'\cdot(\vec{p}'+x\vec{k}')g_{\mu\nu}\Big]\cdot\delta_{-2\vec{p}',\vec{k}'-\vec{k}} \\ =& \frac{-ig^2}{2(4\pi)^2}\left\{\frac{1}{3}\left[\frac{2}{\epsilon}+\ln 4\pi\mu^2e^{-\gamma_E}\right](g_{\mu\nu}k^2-k_\mu k_\nu-3\vec{k}.\vec{k}'g_{\mu\nu})-\int_0^1 dx s(x)\ln\Delta\right\}\end{aligned} \tag{2.20}$$

with

$$s(x) = 2(1-3x+2x^2)(k^2-\vec{k}'^2)g_{\mu\nu}-(1-2x)^2k_\mu k_\nu+4(x-1)(\vec{k}/2+(x-1/2)\vec{k}')^2g_{\mu\nu}. \tag{2.21}$$

Therefore, to cancel the one-loop divergence of the brane correction, brane-localised gauge kinetic terms containing the derivatives with respect to the extra dimensions are required. The remaining integral over x is finite. In conclusion, a bulk scalar in 6D leads to both bulk higher derivative and brane-localised gauge kinetic terms.

2.3 A hypermultiplet contribution

We consider the contribution of a hypermultiplet to the vacuum polarisation. A hypermultiplet is composed of one Dirac fermion and two complex scalars with opposite charges. Using eqs. (2.9) and (2.17) with (2.18), we easily obtain the contribution in a simple form as

$$\begin{aligned}\Pi^{\rm hyper}_{\mu\nu} &= \Pi^f_{\mu\nu}+\Pi^s_{\mu\nu,+}+\Pi^s_{\mu\nu,-}\\ &= -g^2\delta_{\vec{k},\vec{k}'}[(k^2-\vec{k}'^2)g_{\mu\nu}-k_\mu k_\nu]\mu^{4-d}\sum_{\vec{p}'}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-\Delta)^2}.\end{aligned} \tag{2.22}$$

As indicated, the scalars take opposite $\mathbb{Z}_2$ parities. Consequently, we note that the would-be mass corrections to Kaluza-Klein modes of gauge bosons that we referred to earlier

[6] This is particularly relevant in non-supersymmetric models, where similar corrections are present.

in the scalar and fermionic contributions are cancelled out due to supersymmetry. Also the two would-be brane contributions of the scalars are cancelled out. The above result obtained in the component field formalism is in agreement with that obtained in a similar calculation using instead the superfield approach [9].

The explicit evaluation of $\Pi^{\rm hyper}$ is rather technical and we provide the details in appendix D. Essentially one performs the momentum integral in (2.22) in the DR scheme, then re-writes that result in proper-time representation and finally performs the double sum over the discrete momenta $\vec{p} \equiv (p_5, p_6)$. Using eqs. (D.20), (D.21) for $\mathcal{J}_0$, with $a_1 \equiv 1/R_5^2$ and $a_2 \equiv 1/R_6^2$, one finds the contribution of a hypermultiplet in Euclidean space[7]:

$$\begin{aligned}\Pi^{\rm hyper}(k,\vec{k}') &= \frac{i\,\mu^{4-d}}{(4\pi)^{d/2}}\sum_{\vec{p}'}\int_0^1 dx\,\Gamma[2-d/2]\Big[x(1-x)(k^2+\vec{k}'^2)+(\vec{p}'+x\vec{k}')^2\Big]^{d/2-2}\\ &= \frac{i\pi^2\mu^{4-d}}{(2\pi)^d}\sum_{\vec{p}'}\int_0^1 dx\int_0^1\frac{dt}{t^{d/2-1}}\,e^{-\pi t\left[x(1-x)(k^2+\vec{k}'^2)+(\vec{p}'+x\vec{k}')^2\right]}\\ &= \frac{i\pi^2\sigma\,\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\;\mathcal{J}_0\Big[x(1-x)(k^2+\vec{k}'^2);\,xk_5'R_5,\,xk_6'R_6\Big]\\ &= \frac{i\sigma}{(4\pi)^2}\left\{\frac{\pi R_5R_6}{6}(k^2+\vec{k}'^2)\left[\frac{-2}{\epsilon}-\ln 4\pi^2\mu^2\right]+\int_0^1 dx\mathcal{J}_0^{\rm finite}\right]\right\}\end{aligned} \tag{2.23}$$

with

$$\mathcal{J}_0^{\rm finite}[c;c_1,c_2] \equiv \mathcal{J}_0[c;c_1,c_2]-\pi R_5R_6\,c\left(-\frac{2}{\epsilon}\right). \tag{2.24}$$

The above definition of $\mathcal{J}_0^{\rm finite}$ together with (D.20), (D.21) shows that $\mathcal{J}_0^{\rm finite}$ contains no pole in ϵ. Here $c = x(1-x)(k^2+\vec{k}'^2)$, $c_1 = xk_5'R_5$, $c_2 = xk_6'R_6$.

Eq. (2.23) is an important result of this paper. The presence of the momentum-dependent divergence $(k^2+\vec{k}'^2)/\epsilon$ in $\Pi^{\rm hyper}(k,\vec{k}')$ suggests the need for a higher derivative operator as a counterterm to the one-loop correction. Note that the counterterm required is actually a bulk operator since it is of 6D Lorentz invariant form. Its form is the supersymmetric version of that already encountered for bulk scalar and fermion contributions. The need for such an operator is ultimately a reflection of the fact that the initial theory is non-renormalisable. The divergence found is due to re-summing the *infinitely* many bulk mode contributions in $\mathcal{J}_0$, each of them bringing a pole $1/\epsilon$, to obtain instead a k^2/ϵ pole. This means the k^2/ϵ pole is of *non-perturbative* origin. Note that calculations in the past, performed for vanishing external momenta, $k^2+\vec{k}'^2=0$, missed the presence of such higher derivative operators, since the coefficient of the pole is then formally[8] set to zero.

[7]The term $\ln\mu^2$ is made dimensionless by additional logarithmic terms in $\mathcal{J}_0^{\rm finite}$, not shown explicitly.

[8]Strictly speaking this should not be the case: even in such limiting cases, mathematical consistency would require one to introduce an infrared regulator $\lambda_{\rm IR}$ (here replaced by $(k^2+\vec{k}'^2)$) to find a term which "mixes" the IR (λ_{IR}) and UV (ϵ) regulators/terms; such unwelcome UV-IR mixing [11, 14] would signal

If one also introduces a non-trivial complex structure for the underlying torus, $U = R_6/R_5 e^{i\theta}$ (in our case $\theta = \pi/2$), then the coefficient of the pole in eq. (2.23) becomes proportional to $R_5 R_6 \sin\theta$. For $\theta = 0$, when the two dimensions collapse onto each other, one obtains the 5D limit [7] as expected, and no pole is present anymore in that case. This is consistent with the fact that such operators are not generated by one-loop gauge corrections in the 5D case where only a single sum over modes is present. However, at two loop order, two sums over the modes are present and higher derivative operators will again be generated, even in 5D. In conclusion such higher derivative operators are usually present in compactifications, being dynamically generated at the loop level. These operators can also be boundary-localised, in the case of localised superpotential interactions [5 − 7].

Returning to eq. (2.23), the integral over x contains no poles and can be evaluated numerically, using our detailed expressions for $\mathcal{J}_0$ in appendix D. In specific cases further simplifications can occur, for example when $\vec{k}' = 0$. The analysis of the higher derivative operator and of $\Pi^{\rm hyper}$ will be further extended to the case of non-Abelian theories, where its expression and properties will be discussed in greater detail.

3. The effective action for non-Abelian gauge theories on orbifolds

So far we have considered the case of Abelian gauge theories. In this section we continue our analysis of one-loop corrections and derive the effective action for a non-Abelian gauge theory in higher dimensions by developing an approach outlined by Peskin and Schroeder [24]. To this purpose we employ a background field method applicable to orbifold compactifications. First we present the method and derive the general form of the one-loop effective action, then we apply it to the case of the $T^2/\mathbb{Z}_2$ orbifold.

3.1 Background field method for gauge theories in higher dimensions

Let us start with the relevant terms of the 6D supersymmetric action with a hypermultiplet in a representation of the bulk gauge group

$$S = \int d^6X \left[\frac{1}{g^2}\mathrm{Tr}\Big(-\frac{1}{2}F_{MN}F^{MN} + 2\bar{\lambda}\, i\gamma^M D_M\lambda\Big) + \bar{\psi}\, i\bar{\gamma}^M D_M\psi + \sum_{\pm} |D_M\phi_\pm|^2\right] \quad (3.1)$$

where $F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N]$, $D_M\lambda = \partial_M\lambda - i[A_M,\lambda]$, $D_M\psi = (\partial_M - iA_M)\psi$ and $D_M\phi_\pm = (\partial_M \mp iA_M)\phi_\pm$. To introduce the background field method, we split the gauge field into a classical background and a quantum fluctuation:

$$A^a_M \to A^a_M + \mathcal{A}^a_M. \quad (3.2)$$

Then,

$$\bar{\psi} i\bar{\gamma}^M D_M\psi \to \bar{\psi} i\bar{\gamma}^M D_M\psi + \mathcal{A}^a_M \bar{\psi}\bar{\gamma}^M t^a\psi, \quad (3.3)$$

where D_M is the covariant derivative with respect to the background gauge field.

a non-decoupling of high scale physics from its IR region. This would lead one to conclude that higher derivative counterterms are required, if one remembers that the IR regulator can be equivalently replaced by non-zero momentum inflow.

Likewise, the gauge field strength is decomposed as

$$F^a_{MN} \to F^a_{MN} + D_M \mathcal{A}^a_N - D_N \mathcal{A}^a_M + f^{abc}\mathcal{A}^b_M \mathcal{A}^c_N. \tag{3.4}$$

Considering the higher dimensional generalisation of the Faddeev-Popov procedure for the gauge-fixing, the 6D Lagrangian in the Feynman-'t Hooft gauge is given by

$$\begin{aligned}
\mathcal{L}_{FP} = &-\frac{1}{4g^2}\Big(F^a_{MN} + D_M \mathcal{A}^a_N - D_N \mathcal{A}^a_M + f^{abc}\mathcal{A}^b_M \mathcal{A}^c_N\Big)^2 - \frac{1}{2g^2}(D^M \mathcal{A}^a_M)^2 \\
&+\frac{1}{g^2}\Big[2\mathrm{Tr}\Big(\bar\lambda i\gamma^M D_M \lambda\Big) + i\bar\lambda^a f^{abc}\mathcal{A}^b_M \gamma^M \lambda^c\Big] + \bar\psi\Big(i\bar\gamma^M D_M + \mathcal{A}^a_M \bar\gamma^M t^a\Big)\psi \\
&+\sum_{\pm}\Big(|D_M\phi_\pm|^2 \mp (D^M\phi_\pm)^* i\mathcal{A}^a_M t^a \phi_\pm \pm i\phi^*_\pm \mathcal{A}^{aM} t^a D_M \phi_\pm + \phi^*_\pm (\mathcal{A}^a_M t^a)^2 \phi_\pm\Big) \\
&+\bar c^a\Big(-(D^2)^{ac} - D^M f^{abc}\mathcal{A}^b_M\Big)c^c,
\end{aligned} \tag{3.5}$$

where c^a are ghost fields and $D^2 = D_M D^M$.

In order to compute the effective action at one-loop order, we shall ignore terms linear in $\mathcal{A}^a_M$ and integrate over the terms which are quadratic in the gauge fields $\mathcal{A}^a_M$, gauginos λ, hyperinos ψ, hyperscalars ϕ and ghost fields c. After integration by parts, the quadratic terms in $\mathcal{A}^a_M$ are simplified to

$$\mathcal{L}_{\mathcal{A}} = -\frac{1}{2g^2}\Big\{\mathcal{A}^a_M\Big[-(D^2)^{ac}g^{MN} - 2f^{abc}F^{bMN}\Big]\mathcal{A}^c_N\Big\}. \tag{3.6}$$

By using the generator of 6D Lorentz transformations on 6-vectors,

$$\Big(\mathcal{J}^{PQ}\Big)_{MN} = i\Big(\delta^P_M\delta^Q_N - \delta^Q_M\delta^P_N\Big) \tag{3.7}$$

satisfying

$$\mathrm{tr}\Big(\mathcal{J}^{PQ}\mathcal{J}^{MN}\Big) = 2\Big(g^{PM}g^{QN} - g^{PN}g^{QM}\Big), \tag{3.8}$$

we can rewrite the above Lagrangian as

$$\mathcal{L}_{\mathcal{A}} = -\frac{1}{2g^2}\Big\{\mathcal{A}^a_M\Big[-(D^2)^{ac}g^{MN} + 2\left(\frac{1}{2}F^b_{PQ}\mathcal{J}^{PQ}\right)^{MN}(t^b_G)^{ac}\Big]\mathcal{A}^c_N\Big\} \tag{3.9}$$

with $(t^b_G)^{ac} \equiv if^{abc}$. Further, the quadratic terms in fermion fields are

$$\mathcal{L}_f = \frac{1}{g^2}\mathrm{Tr}\Big(2\bar\lambda i\gamma^M D_M\lambda\Big) + \bar\psi i\bar\gamma^M D_M\psi. \tag{3.10}$$

Integrating over the fermion fields, we obtain the functional determinant of the operator $(i\gamma^M D_M)$ for the gaugino and $(i\bar\gamma^M D_M)$ for the hyperino. Finally, the quadratic terms in hyperscalars ($\mathcal{L}_s$) and ghost fields ($\mathcal{L}_g$) are

$$\mathcal{L}_s = \sum_{\pm}(\phi^a_\pm)^*[-(D^2)^{ac}]\phi^c_\pm, \tag{3.11}$$

$$\mathcal{L}_g = \bar c^a[-(D^2)^{ac}]c^c. \tag{3.12}$$

With these findings, after performing the path integral for the terms quadratic in quantum fluctuations, we obtain the effective action for the classical field A^a_M at one-loop order as

$$
\begin{aligned}
e^{i\Gamma[A]} &= \exp\Big[i\int d^6X\Big(-\frac{1}{4g^2}(F^a_{MN})^2+\mathcal{L}_{\text{c.t.}}\Big)\Big] \qquad (3.13)\\
&\times(\det\Delta_{G,1})^{-\frac{1}{2}}(\det\mathcal{D}_G)^{+1}[\det(-\Delta_{G,0})]^{+1}(\det\mathcal{D}_r)^{+1}[\det(-\Delta_{r,0})]^{-1}[\det(-\Delta_{r^*,0})]^{-1}
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta_{G,1} &= \frac{1}{g^2}\Big[\Big(-D_1^2 g^{MN}+2\,\big(\frac{1}{2}F^b_{PQ1}\mathcal{J}^{PQ}\big)^{MN}t^b_G\Big)\,\delta^{\mathcal{A}_N}_{12}\Big],\\
\Delta_{G,0} &= -D_1^2\,\delta^c_{12}, \qquad \Delta_{r,0} = -D_1^2\,\delta^{\phi_r}_{12},\\
\mathcal{D}_G &= \frac{1}{g^2}\Big(i\gamma^M\partial_{M1}+A^a_{M1}t^a_G\gamma^M\Big)\,\delta^\lambda_{12},\\
\mathcal{D}_r &= \Big(i\bar{\gamma}^M\partial_{M1}+A^a_{M1}t^a_r\bar{\gamma}^M\Big)\,\delta^\psi_{12}, \qquad (3.14)
\end{aligned}
$$

where r denotes the corresponding representation and an extra index "1" as in f_1 denotes $f(X_1)$ while the δ_{12}'s are defined as functional differentiations presented below. Finally, as the upper letter on the δ_{12}'s imply, the above expressions are contributions of the gauge bosons, ghosts, hyperscalars, gaugino and hyperino fields respectively. Further

$$
(\delta^{\mathcal{A}_M}_{12})^a{}_b \equiv \frac{\delta\mathcal{A}^a_M(X_1)}{\delta\mathcal{A}^b_M(X_2)}, \qquad (\delta^{\phi_r}_{12})^a{}_b \equiv \frac{\delta\phi^a_r(X_1)}{\delta\phi^b_r(X_2)}, \qquad (3.15)
$$

and similar for the remaining fields. Note that as long as there is no orbifold action present $\delta^{\mathcal{A}_M}_{12}=\delta^{\phi_r}_{12}=\delta^\lambda_{12}=\delta^\psi_{12}=\delta^6(X_1-X_2)$. With these observations, we have the full one-loop effective action

$$
\begin{aligned}
\Gamma[A] = \int d^6X\Big(&-\frac{1}{4g^2}(F^a_{MN})^2+\mathcal{L}_{\text{c.t.}}\Big)\\
&+\frac{i}{2}\Big[\ln\det\Delta_{G,1}-2\,\ln\det\mathcal{D}_G-2\,\ln\det(-\Delta_{G,0})\\
&\qquad -2\,\ln\det\mathcal{D}_r+2\,\ln\det(-\Delta_{r,0})+2\,\ln\det(-\Delta_{r^*,0})\Big]. \qquad (3.16)
\end{aligned}
$$

This is the general formula for the one-loop effective action in higher dimensions with our field content. It can be applied to specific cases, by computing the above determinants, after specifying the boundary conditions for the fields involved.

3.2 The effective action on the $T^2/\mathbb{Z}_2$ orbifold

We can now apply the method presented in the previous section to the case of orbifold compactifications, where important changes appear due to the presence of the associated boundary conditions with respect to the compact dimensions. On the orbifold $T^2/\mathbb{Z}_2$, the orbifold boundary conditions are given by

$$\begin{aligned} \mathcal{A}^a_\mu(x,-z) &= \mathcal{A}^a_\mu(x,z), & \mathcal{A}^a_{5,6}(x,-z) &= -\mathcal{A}^a_{5,6}(x,z),\\ c^a(x,-z) &= c^a(x,z), & \lambda(x,-z) &= i\gamma^5\,\lambda(x,z),\\ \psi(x,-z) &= i\gamma^5\,\eta\,\psi(x,z), & \phi_\pm(x,-z) &= \pm\eta\phi_\pm(x,z) \end{aligned} \tag{3.17}$$

where η can be chosen either $+1$ or -1. Taking into account these boundary conditions, the functional differentiations defined in (3.15) can be made orbifold-compatible as follows:

$$\begin{aligned} \delta^{\mathcal{A}_\mu}_{12} &= \frac{1}{2}\Big(\delta^6(X_1-X_2)+\delta^6(X_1+X_2)\Big) = \delta^c_{12} \equiv \delta^+_{12},\\ \delta^{\mathcal{A}_n}_{12} &= \frac{1}{2}\Big(\delta^6(X_1-X_2)-\delta^6(X_1+X_2)\Big) \equiv \delta^-_{12},\\ \delta^{\phi_\pm}_{12} &= \frac{1}{2}\Big(\delta^6(X_1-X_2)\pm\eta\,\delta^6(X_1+X_2)\Big)\\ \delta^{\lambda}_{12} &= \frac{1}{2}\Big(\delta^6(X_1-X_2)-i\gamma^5\delta^6(X_1+X_2)\Big)\\ \delta^{\psi}_{12} &= \frac{1}{2}\Big(\delta^6(X_1-X_2)-i\eta\,\gamma^5\delta^6(X_1+X_2)\Big) \end{aligned} \tag{3.18}$$

where $\delta^6(X_1\pm X_2)\equiv\delta^4(x_1-x_2)\delta^2(z_1\pm z_2)$. We can now evaluate the determinants in (3.16) giving the contributions of various fields to the one-loop effective action. To second order in the background gauge field we have from eq. (3.16)

$$\begin{aligned} \Gamma^{(2)}[A_M] = {}& \frac{1}{2g^2}\sum_{\vec{k}}\int\frac{d^4k}{(2\pi)^4}A^a_M(-k,-\vec{k})A^b_N(k,\vec{k})(-(k^2-\vec{k}^2)g^{MN}+k^Mk^N)\\ &+\frac{i}{2}\Big[W_{G,1}-2W_{G,0}-2W_{\rm gaugino}+2W_{\rm hypers}-2W_{\rm hyperino}\Big] \end{aligned} \tag{3.19}$$

where each W is the quadratic term of the corresponding log determinant in (3.16).

3.2.1 Gauge field contribution $W_{G,1}$

We start with the contribution of the gauge bosons and first introduce the notation:

$$\mathcal{M}\equiv\begin{pmatrix} -\partial_1^2 g^{\mu\nu}\delta^+_{12} & 0\\ 0 & -\partial_1^2 g^{mn}\delta^-_{12}\end{pmatrix},\quad \mathcal{N}\equiv\begin{pmatrix} (\Delta_G g^{\mu\nu}+\Delta^{\mu\nu})_1\delta^+_{12} & \Delta^{\mu n}_1\delta^-_{12}\\ \Delta^{m\nu}_1\delta^+_{12} & (\Delta_G g^{mn}+\Delta^{mn})_1\delta^-_{12}\end{pmatrix} \tag{3.20}$$

where

$$\begin{aligned} \Delta_G &\equiv \Delta^{(1)}_G+\Delta^{(2)}_G\\ \Delta^{(1)}_G &\equiv i\Big[\partial^M A^a_M t^a_G + A^a_M t^a_G\partial^M\Big], \qquad \Delta^{(2)}_G \equiv A^{aM}t^a_G A^b_M t^b_G,\\ \Delta^{MN} &\equiv 2\left(\frac{1}{2}F^b_{PQ}\mathcal{J}^{PQ}\right)^{MN} t^b_G. \end{aligned} \tag{3.21}$$

With this notation and (3.14) we obtain

$$\begin{aligned}\ln\det\Delta_{G,1} &= \ln\det\frac{1}{g^2}\Big[\mathcal{M}+\mathcal{N}\Big] = \ln\det\frac{1}{g^2}\mathcal{M} - \sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{tr}\Big[(\mathcal{O}_M{}^N)^n\Big]\\ &= \ln\det\frac{1}{g^2}\mathcal{M} - \mathrm{tr}(\mathcal{O}_\nu{}^\mu) - \mathrm{tr}(\mathcal{O}_m{}^n)\\ &-\frac{1}{2}\Big[\mathrm{tr}(\mathcal{O}_\nu{}^\lambda\mathcal{O}_\lambda{}^\mu)+\mathrm{tr}(\mathcal{O}_m{}^l\mathcal{O}_l{}^n)+\mathrm{tr}(\mathcal{O}_\nu{}^l\mathcal{O}_l{}^\nu)+\mathrm{tr}(\mathcal{O}_m{}^\lambda\mathcal{O}_\lambda{}^n)\Big]+\cdots,\end{aligned} \tag{3.22}$$

where we introduced

$$\mathcal{O}_M{}^N \equiv \begin{pmatrix} \delta^+_{12} i(-\partial_2^2)^{-1} g_{\nu\lambda} & 0 \\ 0 & \delta^-_{12} i(-\partial_2^2)^{-1} g_{ml}\end{pmatrix}\begin{pmatrix} i(\Delta_G g^{\lambda\mu}+\Delta^{\lambda\mu})_2\delta^+_{23} & i\Delta_2^{\lambda n}\delta^-_{23} \\ i\Delta_2^{l\mu}\delta^+_{23} & i(\Delta_G g^{ln}+\Delta^{ln})_2\delta^-_{23}\end{pmatrix} \tag{3.23}$$

Therefore, the terms in $\ln\det\Delta_{G,1}$ quadratic in the background gauge field are

$$W_{G,1}[A_M] = 4\,(T_1^{G+}+T_2^{G+}) + 2\,(T_1^{G-}+T_2^{G-}) + T_3^G + T_4^G + T_5^G + T_6^G. \tag{3.24}$$

Their origin is as follows: $4(T_1^{G+}+T_2^{G+})$ accounts for part of the term $\mathrm{tr}(\mathcal{O}_\nu{}^\lambda\mathcal{O}_\lambda{}^\mu)$ and for the term $\mathrm{tr}(\mathcal{O}_\nu{}^\mu)$, while $2(T_1^{G-}+T_2^{G-})$ accounts for similar terms but with matrices entries with extra dimensional Lorentz indices. The different factors multiplying them (4 and 2) arise from the different metric contractions. Further, T_3^G accounts for (the remaining part of) $\mathrm{tr}(\mathcal{O}_\nu{}^\lambda\mathcal{O}_\lambda{}^\mu)$ while T_4^G accounts for similar contribution but with all Lorentz indices extra dimensional. Finally, $T_{5,6}^G$ account for the "mixed" indices contributions, the last two terms in the last line of (3.22), respectively. All these contributions can be easily identified by recalling that δ^+_{ij} (δ^-_{ij}) arise with contributions from 4D (extra dimensional) Lorentz indices, respectively, as seen from the definition of $\mathcal{O}_M{}^N$. The results of evaluating the terms in (3.24) are then

$$\begin{aligned}T_1^{G\pm}+T_2^{G\pm} &\equiv -\frac{1}{2}\,\mathrm{tr}\Big[\Big(\delta^\pm_{12}\,i(-\partial_2^2)^{-1}\,(i\Delta^{(1)}_{G,2}\,\delta^\pm_{23})\Big)\,\Big(\delta^\pm_{34} i(-\partial_4^2)^{-1}\,(i\Delta^{(1)}_{G,4}\,\delta^\pm_{41})\Big)\Big]\\ &\quad -\mathrm{tr}\Big[\delta^\pm_{12} i(-\partial_2^2)^{-1}(i\Delta^{(2)}_{G,2}\delta^\pm_{21})\Big]\\ &= -\frac{1}{2}\,C_2(G)\sum_{\vec{k},\vec{k}'}\int\frac{d^4k}{(2\pi)^4}A^{aM}(-k,-\vec{k}')\,A^{aN}(k,\vec{k})\;\Pi^s_{MN,\pm}.\end{aligned} \tag{3.25}$$

One should consider in (3.25) either the upper or the lower signs only. Further T_3^G is generated by parity-even gauge fields, as the presence of δ^+_{ij} shows and equals

$$
\begin{aligned}
T_3^G &\equiv -\frac{1}{2}\mathrm{tr}\Big[\Big(\delta^+_{12}\, i(-\partial_2^2)^{-1}\,(i(\Delta_\nu{}^\lambda)_2\,\delta^+_{23})\Big)\,\Big(\delta^+_{34} i(-\partial_4^2)^{-1}\,(i(\Delta_\lambda{}^\mu)_4\,\delta^+_{41})\Big)\Big] \\
&= 2\,\mathrm{tr}\Big[\mathcal{J}^{\rho\sigma} t^a_G \mathcal{J}^{\alpha\beta} t^b_G\Big] \sum_{\vec{k},\vec{k}'} \int \frac{d^4k}{(2\pi)^4}\; A^a_\mu(-k,-\vec{k}')\, A^b_\nu(k,\vec{k})\, k_\rho\, g^\mu_\sigma\, k_\alpha\, g^\nu_\beta \\
&\quad \times \sum_{\vec{p},\vec{p}'} \int \frac{d^4p}{(2\pi)^4}\; \tilde{G}_+(p,\vec{p},\vec{p}')\; \tilde{G}_+(p+k,\vec{p}'+\vec{k}',\vec{p}+\vec{k}) \\
&= 4C_2(G) \sum_{\vec{k},\vec{k}'} \int \frac{d^4k}{(2\pi)^4} A^a_\mu(-k,-\vec{k}')\, A^a_\nu(k,\vec{k}) \Big(k^2 g^{\mu\nu} - k^\mu k^\nu\Big)\, \Pi^G_{++}, \qquad (3.26)
\end{aligned}
$$

T_4^G has similar form, but involves only parity-odd fields (notice the presence of δ^-_{ij}):

$$
\begin{aligned}
T_4^G &\equiv -\frac{1}{2}\mathrm{tr}\Big[\Big(\delta^-_{12}\, i(-\partial_2^2)^{-1}\,(i(\Delta_m{}^l)_2\,\delta^-_{23})\Big)\,\Big(\delta^-_{34} i(-\partial_4^2)^{-1}\,(i(\Delta_l{}^n)_4\,\delta^-_{41})\Big)\Big] \\
&= 2\mathrm{tr}\Big[\mathcal{J}^{ij} t^a_G \mathcal{J}^{kl} t^b_G\Big] \sum_{\vec{k},\vec{k}'} \int \frac{d^4k}{(2\pi)^4}\; A^a_m(-k,-\vec{k}')\, A^b_n(k,\vec{k})\, k'_i\, g^m_j\, k_k\, g^n_l \\
&\quad \times \sum_{\vec{p},\vec{p}'} \int \frac{d^4p}{(2\pi)^4}\; \tilde{G}_-(p,\vec{p},\vec{p}')\; \tilde{G}_-(p+k,\vec{p}'+\vec{k}',\vec{p}+\vec{k}), \\
&= 4C_2(G) \sum_{\vec{k},\vec{k}'} \int \frac{d^4k}{(2\pi)^4} A^a_m(-k,-\vec{k}')\, A^a_n(k,\vec{k}) \Big(-\vec{k}'\cdot\vec{k} g^{mn} - k^m k'^n\Big)\, \Pi^G_{--}, \qquad (3.27)
\end{aligned}
$$

Finally T_5^G and T_6^G have similar structure, involving parity-odd and -even component fields:

$$
\begin{aligned}
T_5^G &\equiv -\frac{1}{2}\mathrm{tr}\Big[\Big(\delta^+_{12}\, i(-\partial_2^2)^{-1}\,(i(\Delta_\nu{}^l)_2\,\delta^-_{23})\Big)\,\Big(\delta^-_{34}\, i(-\partial_4^2)^{-1}\,(i(\Delta_l{}^\mu)_4\,\delta^+_{41})\Big)\Big] \\
&= 2\,\mathrm{tr}\Big[(\mathcal{J}^{\lambda k})_\nu{}^l t^a_G (\mathcal{J}^{\rho n})_l{}^\mu t^b_G\Big] \Big(k_\lambda A^a_k(-k,-\vec{k}') - k'_k A^a_\lambda(-k,-\vec{k}')\Big) \\
&\quad \Big(k_\rho A^b_n(k,\vec{k}) - k_n A^b_\rho(k,\vec{k})\Big) \sum_{\vec{p},\vec{p}'} \int \frac{d^4p}{(2\pi)^4}\; \tilde{G}_-(p,\vec{p},\vec{p}')\; \tilde{G}_+(p+k,\vec{p}'+\vec{k}',\vec{p}+\vec{k}) \\
&= -2C_2(G) \sum_{\vec{k},\vec{k}'} \int \frac{d^4k}{(2\pi)^4} \Big(k_\mu A^a_k(-k,-\vec{k}') - k'_k A^a_\mu(-k,-\vec{k}')\Big) \\
&\qquad \times \Big(k_\rho A^a_n(k,\vec{k}) - k_n A^a_\rho(k,\vec{k})\Big)\, g^{\rho\mu} g^{kn}\, \Pi^G_{-+}, \qquad (3.28)
\end{aligned}
$$

and

$$T_6^G \equiv -\frac{1}{2}\mathrm{tr}\Big[\Big(\delta_{12}^-\, i(-\partial_2^2)^{-1}\,(i(\Delta_n{}^{\lambda})_2\,\delta_{23}^+)\Big)\,\Big(\delta_{34}^+\, i(-\partial_4^2)^{-1}\,(i(\Delta_\lambda{}^{m})_4\,\delta_{41}^-)\Big)\Big]$$

$$= 2\,\mathrm{tr}\Big[(\mathcal{J}^{\mu k})_n{}^{\lambda} t_G^a (\mathcal{J}^{\rho l})_\lambda{}^{m} t_G^b\Big] \sum_{\vec{k},\vec{k}'} \int \frac{d^4k}{(2\pi)^4}\Big(k_\mu A_k^a(-k,-\vec{k}') - k'_k A_\mu^a(-k,-\vec{k}')\Big)$$

$$\Big(k_\rho A_l^b(k,\vec{k}) - k_l A_\rho^b(k,\vec{k})\Big) \sum_{\vec{p},\vec{p}'} \int \frac{d^4p}{(2\pi)^4}\; \tilde{G}_+(p,\vec{p},\vec{p}')\; \tilde{G}_-(p+k,\vec{p}'+\vec{k}',\vec{p}+\vec{k})$$

$$= -2C_2(G) \sum_{\vec{k},\vec{k}'} \int \frac{d^4k}{(2\pi)^4}\Big(k_\mu A_k^a(-k,-\vec{k}') - k'_k A_\mu^a(-k,-\vec{k}')\Big)$$

$$\times\Big(k_\rho A_n^a(k,\vec{k}) - k_n A_\rho^a(k,\vec{k})\Big)\, g^{\rho\mu} g^{kn}\, \Pi_{+-}^G. \tag{3.29}$$

In the equations above we used the notation $C_2(G)$ defined by $\mathrm{tr}(t_G^a t_G^b) = C_2(G)\delta^{ab}$. In terms of the bulk propagator for bosons (See also eq. (B.7)),

$$\tilde{G}_\pm(p,\vec{p},\vec{p}') = \frac{i}{2}\frac{\delta_{\vec{p},\vec{p}'} \pm \delta_{\vec{p},-\vec{p}'}}{p^2 - \vec{p}^2}, \tag{3.30}$$

one has the following expressions for $\Pi_{MN,\pm}^s$ and $\Pi_{\alpha\beta}^G$ used previously

$$\Pi_{MN,\pm}^s = \sum_{\vec{p},\vec{p}'} \int \frac{d^4p}{(2\pi)^4}\Big[-(2p'+k')_M(2p+k)_N \tilde{G}_\pm(p+k,\vec{p}'+\vec{k}',\vec{p}+\vec{k})$$

$$+2ig_{MN}\delta_{\vec{p}',\vec{p}+\vec{k}-\vec{k}'}\Big]\cdot \tilde{G}_\pm(p,\vec{p},\vec{p}')$$

$$= -\frac{1}{2}\sum_{\vec{p}'} \int \frac{d^4p}{(2\pi)^4} \frac{-(2p'+k')_M(2p+k)_N + 2g_{MN}[(p+k)^2 - (\vec{p}'+\vec{k}')^2]}{(p^2-\vec{p}'^2)[(p+k)^2 - (\vec{p}'+\vec{k}')^2]}$$

$$\times\Big(\delta_{\vec{k},\vec{k}'} \pm \delta_{-2\vec{p}',\vec{k}'-\vec{k}}\Big), \tag{3.31}$$

$$\Pi_{\pm\pm}^G = \sum_{\vec{p},\vec{p}'} \int \frac{d^4p}{(2\pi)^4}\; \tilde{G}_\pm(p,\vec{p},\vec{p}')\, \tilde{G}_\pm(p+k,\vec{p}+\vec{k},\vec{p}'+\vec{k}')$$

$$= -\frac{1}{2}\sum_{\vec{p}'} \int \frac{d^4p}{(2\pi)^4} \frac{\delta_{\vec{k},\vec{k}'} \pm \delta_{-2\vec{p}',\vec{k}'-\vec{k}}}{(p^2-\vec{p}'^2)[(p+k)^2 - (\vec{p}'+\vec{k}')^2]} \tag{3.32}$$

and

$$\Pi_{\pm\mp}^G = \sum_{\vec{p},\vec{p}'} \int \frac{d^4p}{(2\pi)^4}\; \tilde{G}_\pm(p,\vec{p},\vec{p}')\, \tilde{G}_\mp(p+k,\vec{p}+\vec{k},\vec{p}'+\vec{k}') = \Pi_{\pm,\pm}^G. \tag{3.33}$$

To obtain the above results for T_5^G and T_6^G we had to change the order of operators in an appropriate way, by using $\mathcal{O}_2\delta_{23}^\pm = \delta_{23}^\pm\mathcal{O}_3$ for the $\mathbb{Z}_2$-even operator $\mathcal{O}$ while $\widetilde{\mathcal{O}}_2\delta_{23}^\pm = \delta_{23}^\mp\widetilde{\mathcal{O}}_3$ for the $\mathbb{Z}_2$-odd operator $\widetilde{\mathcal{O}}$. Further, to simplify the Kronecker deltas, we have taken into account the $\mathbb{Z}_2$-parity conditions: $A_\mu^a(k,\vec{k}') = A_\mu^a(k,-\vec{k}')$ and $A_m^a(k,\vec{k}') = -A_m^a(k,-\vec{k}')$. This concludes the evaluation of the gauge fields contribution $W_{G,1}$ of (3.24).

3.2.2 Ghost field contribution $W_{G,0}$

Next we evaluate the determinant of the ghost field contribution (3.14) with (3.18)

$$\begin{aligned}\ln\det(-\Delta_{G,0}) &= \ln\det\Big((\partial^2-\Delta_G)_1\delta^+_{12}\Big)\\ &= \ln\det(\partial_1^2\delta^+_{12}) - \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}\Big[\Big(\delta^+_{12}i(-\partial_2^2)^{-1}i(\Delta_G)_2\delta^+_{23}\Big)^n\Big]. \qquad (3.34)\end{aligned}$$

from which, upon expansion, we isolate the quadratic terms for the background field as

$$W_{G,0}[A_M] = T_1^{G+} + T_2^{G+}. \qquad (3.35)$$

The sum on the right-hand side was already computed in (3.25).

3.2.3 Hyperscalar contribution $W_{\mathbf{hypers}}$

Likewise, the quadratic terms from the determinant for hyperscalars are, with (3.14), (3.18)

$$\begin{aligned}\ln\det(-\Delta_{r,0}) &= \ln\det\Big((\partial^2-\Delta_r)_1\delta^\eta_{12}\Big)\\ &= \ln\det(\partial_1^2\delta^\eta_{12}) - \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}\Big[\Big(\delta^\eta_{12}i(-\partial_2^2)^{-1}i(\Delta_r)_2\delta^\eta_{23}\Big)^n\Big]. \qquad (3.36)\end{aligned}$$

with the notation of Δ as in eq. (3.21) with $G\to r$. One finds from (3.36)

$$W_{\text{hypers}}[A_M] = (T_1^{r+}+T_2^{r+}) + (T_1^{r-}+T_2^{r-}) \qquad (3.37)$$

where $T^{r\pm}_{1,2} = \big[C(r)/C_2(G)\big]\,T^{G\pm}_{1,2}$ and with $T_1^{G\pm}+T_2^{G\pm}$ already evaluated in eq. (3.25). Here $C(r)$ is defined by $\mathrm{tr}(t_r^a t_r^b) = C(r)\delta^{ab}$.

3.2.4 Gaugino and hyperino contributions $W_{\mathbf{gaugino}}$ and $W_{\mathbf{hyperino}}$

Finally, we evaluate the determinants for the fermion fields, which are expanded as (using again (3.14), (3.18))

$$\begin{aligned}\ln\det\mathcal{D}_G &= \ln\det\Big[\frac{1}{g^2}(i\gamma^M\partial_{M1} + A^a_{M1}t^a_G\gamma^M)\delta^\lambda_{12}\Big]\\ &= \ln\det\Big[\frac{1}{g^2}i\gamma^M\partial_{M1}\delta^\lambda_{12}\Big] - \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}\Big[\Big\{\delta^\lambda_{12}\frac{i}{i\gamma^P\partial_{P2}}(iA^a_{M2}t^a_G\gamma^M\delta^\lambda_{23})\Big\}^n\Big], \qquad (3.38)\\ \ln\det\mathcal{D}_r &= \ln\det\Big[(i\bar\gamma^M\partial_{M1} + A^a_{M1}t^a_r\bar\gamma^M)\delta^\psi_{12}\Big]\\ &= \ln\det\Big[i\bar\gamma^M\partial_{M1}\delta^\psi_{12}\Big] - \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}\Big[\Big\{\delta^\psi_{12}\frac{i}{i\bar\gamma^P\partial_{P2}}(iA^a_{M2}t^a_r\bar\gamma^M\delta^\psi_{23})\Big\}^n\Big]. \qquad (3.39)\end{aligned}$$

with the former (latter) for gaugino (hyperino) fields, respectively. From these eqs. the quadratic terms coming from the determinants of gaugino and hyperino are evaluated to

$$\begin{aligned}W_{\text{gaugino}}[A_M] &= -\frac{1}{2}\mathrm{tr}\Big[\delta^\lambda_{12}\frac{i}{i\gamma^P\partial_{P2}}(iA^a_{M2}t^a_G\gamma^M\delta^\lambda_{23})\;\delta^\lambda_{34}\frac{i}{i\gamma^Q\partial_{Q4}}(iA^b_{N4}t^b_G\gamma^N\delta^\lambda_{41})\Big]\\ &= \frac{1}{2}\mathrm{tr}(t^a_G t^b_G)\sum_{\vec k,\vec k'}\int\frac{d^4k}{(2\pi)^4}A^{aM}(-k,-\vec k')A^{bN}(k,\vec k)\,\tilde\Pi^f_{MN}, \qquad (3.40)\end{aligned}$$

$$W_{\text{hyperino}}[A_M] = -\frac{1}{2}\text{tr}\Big[\delta^\psi_{12}\frac{i}{i\bar\gamma^P\partial_{P2}}(iA^a_{M2}t^a_r\bar\gamma^M\delta^\psi_{23})\ \delta^\psi_{34}\frac{i}{i\bar\gamma^Q\partial_{Q4}}(iA^a_{N4}t^a_r\bar\gamma^N\delta^\psi_{41})\Big]$$
$$= \frac{1}{2}\text{tr}(t^a_rt^b_r)\sum_{\vec k,\vec k'}\int\frac{d^4k}{(2\pi)^4}A^{aM}(-k,-\vec k')A^{bN}(k,\vec k)\,\Pi^f_{MN} \tag{3.41}$$

Here we introduced the following self-energies

$$\tilde\Pi^f_{MN} \equiv \sum_{\vec p,\vec p'}\int\frac{d^4p}{(2\pi)^4}\text{Tr}\Big[\tilde D_\lambda(p,\vec p,\vec p')\gamma_M\tilde D_\lambda(p+k,\vec p'+\vec k',\vec p+\vec k)\gamma_N\Big], \tag{3.42}$$

$$\Pi^f_{MN} \equiv \sum_{\vec p,\vec p'}\int\frac{d^4p}{(2\pi)^4}\text{Tr}\Big[\tilde D_\psi(p,\vec p,\vec p')\bar\gamma_M\tilde D_\psi(p+k,\vec p'+\vec k',\vec p+\vec k)\bar\gamma_N\Big], \tag{3.43}$$

and used the propagators on $T^2/\mathbb{Z}_2$ (for details see the appendix, eq. (B.4))

$$\tilde D_\lambda(p,\vec p,\vec p') = \frac{i}{2}\left(\frac{\delta_{\vec p,\vec p'}}{\not p+\gamma_5p_5-p_6}-\frac{\delta_{\vec p,-\vec p'}}{\not p+\gamma_5p_5-p_6}i\gamma_5\right), \tag{3.44}$$

$$\tilde D_\psi(p,\vec p,\vec p') = \frac{i}{2}\left(\frac{\delta_{\vec p,\vec p'}}{\not p+\gamma_5p_5+p_6}-\frac{\eta\delta_{\vec p,-\vec p'}}{\not p+\gamma_5p_5+p_6}i\gamma_5\right). \tag{3.45}$$

This concludes the identification of all component field contributions to the effective action. We now have the necessary technical results eqs. (3.24), (3.35), (3.37), (3.40), (3.41), to analyse the one-loop effective action of non-Abelian gauge theories on $T^2/\mathbb{Z}_2$.

3.2.5 The one-loop effective action on $T^2/\mathbb{Z}_2$, its poles and counterterms

In the following we concentrate on the 4D gauge field part of the effective action. In this case, we note that $\Pi^f_{\mu\nu}$ and $\Pi^s_{\mu\nu,\pm}$ are the same as the ones in (2.9), (2.17), respectively, which were obtained by using the Feynman diagram approach in the U(1) case. Therefore, using (3.19), the 4D gauge field part of the effective action can be written as

$$\Gamma^{(2)}[A_\mu] = \frac{1}{2g^2}\sum_{\vec k}\int\frac{d^4k}{(2\pi)^4}A^a_\mu(-k,-\vec k)\,A^a_\nu(k,\vec k)\Big(-(k^2-\vec k^2)g^{\mu\nu}+k^\mu k^\nu\Big)$$
$$+\frac{i}{2}\sum_{\vec k,\vec k'}\int\frac{d^4k}{(2\pi)^4}A^{a\mu}(-k,-\vec k')\,A^{a\nu}(k,\vec k) \tag{3.46}$$
$$\times\Big\{C_2(G)\Big[-\Pi^{\text{hyper}}_{\mu\nu}+4(k^2g_{\mu\nu}-k_\mu k_\nu)\Pi^G_{++}-2\vec k\cdot\vec k'g_{\mu\nu}(\Pi^G_{+-}+\Pi^G_{-+})\Big]-C(r)\Pi^{\text{hyper}}_{\mu\nu}\Big\}$$

where

$$\Pi^{\text{hyper}}_{\mu\nu} \equiv \Pi^s_{\mu\nu,+}+\Pi^s_{\mu\nu,-}+\Pi^f_{\mu\nu}. \tag{3.47}$$

Then, by decomposing this effective action into bulk and brane parts, we reach the main result of section 3.2:

$$\Gamma^{(2)}[A_\mu] = \Gamma_{\text{bulk}}+\Gamma_{\text{brane}} \tag{3.48}$$

with

$$\Gamma_{\text{bulk}} = \frac{1}{2}\sum_{\vec{k},\vec{k}'}\int \frac{d^4k}{(2\pi)^4} A^a_\mu(-k,-\vec{k}')A^a_\nu(k,\vec{k})\big((k^2-\vec{k}^2)g^{\mu\nu}-k^\mu k^\nu\big)$$
$$\times\left[-\frac{1}{g^2}-i\Big(C_2(G)-C(r)\Big)\,\Pi^{\text{hyper}}(k,\vec{k}')\right]\delta_{\vec{k},\vec{k}'}, \tag{3.49}$$

$$\Gamma_{\text{brane}} = \frac{1}{2}\sum_{\vec{k},\vec{k}'}\int \frac{d^4k}{(2\pi)^4} A^a_\mu(-k,-\vec{k}')A^a_\nu(k,\vec{k})\big(k^2 g^{\mu\nu}-k^\mu k^\nu\big)\Big[-4iC_2(G)\Pi^{\text{local}}(k,\vec{k},\vec{k}')\Big] \tag{3.50}$$

where

$$\Pi^{\text{hyper}}(k,\vec{k}') \equiv \mu^{4-d}\sum_{\vec{p}'}\int \frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-\vec{p}'^2)[(p+k)^2-(\vec{p}'+\vec{k}')^2]}, \tag{3.51}$$

$$\Pi^{\text{local}}(k,\vec{k},\vec{k}') \equiv \frac{\mu^{4-d}}{2}\sum_{\vec{p}'}\int \frac{d^dp}{(2\pi)^d}\frac{\delta_{-2\vec{p}',\vec{k}'-\vec{k}}}{(p^2-(\vec{p}')^2)[(p+k)^2-(\vec{p}'+\vec{k}')^2]}. \tag{3.52}$$

From the expression of Γ_{bulk} we see that the bulk correction comes with the standard beta function coefficient[9] in 6D which is given by $C(r)-C_2(G)$. Note also that, as in the Abelian case discussed previously, a hypermultiplet does not generate a boundary-localised gauge coupling. However, a 6D bulk counterterm can be present as we already saw in the Abelian case (2.23), when evaluating Π^{hyper}. Unlike the hypermultiplet, a vector multiplet does generate boundary-localised gauge couplings, see eqs. (3.50), (3.52). The corresponding (4D) counterterm that we discuss shortly must then be localised at the fixed points.

The divergent nature of Π^{hyper} of eq. (3.51) was already presented and discussed to some extent in the Abelian case, section 2, eq. (2.23). Since Π^{hyper} also appears in the bulk correction in the case of non-Abelian gauge theories, eq. (3.51), we analyse this in further detail. From eq. (2.23), let us recall the following,

$$\Pi^{\text{hyper}}(k,\vec{k}') = \frac{i\sigma}{(4\pi)^2}(2\pi\mu)^\epsilon\int_0^1 dx\,\mathcal{J}_0\Big[x(1-x)(k^2+\vec{k}^{\prime 2});xk'_5R_5,xk'_6R_6\Big]. \tag{3.53}$$

The exact expression of $\mathcal{J}_0$ is needed for studying the finite effects and the dependence of the zero-mode gauge coupling on the momentum k^2. This expression would also be needed to study dimensional crossover effects [15] of the coupling at $k^2\sim 1/R^2_{5,6}$. Since $\mathcal{J}_0$ is rather complicated, we present $\mathcal{J}_0$ below, for a somewhat simpler case of $k'_5=k'_6=0$. From eqs. (D.1), (D.20), (D.21), (D.22) and with the following notations

$$c\equiv x(1-x)k^2,\quad a_1\equiv\frac{1}{R_5^2},\quad a_2\equiv\frac{1}{R_6^2},\quad s_{\tilde{n}_1}\equiv 2\pi\tilde{n}_1\sqrt{\frac{c}{a_1}},\quad \gamma(n_1)\equiv\frac{(c+a_1n_1^2)^{\frac{1}{2}}}{\sqrt{a_2}}, \tag{3.54}$$

one has, if $0\le c/a_1<1$:

[9] Because the number of modes is reduced due to orbifolding, the beta function coefficient is 1/2 times that for a torus compactification.

$$\mathcal{J}_0[c;0,0] = \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon}+\ln\left[4\pi\, a_1\, e^{-\gamma_E}\right]\right] - \sum_{n_1\in\mathbf{Z}} \ln\left|1-e^{-2\pi\,\gamma(n_1)}\right|^2 + \frac{\pi}{3}\sqrt{\frac{a_1}{a_2}} - 2\pi\sqrt{\frac{c}{a_2}}$$

$$-2\,\frac{c\,\pi^{\frac{1}{2}}}{\sqrt{a_1 a_2}} \sum_{p\geq 1} \frac{\Gamma[p+1/2]}{(p+1)!}\left[\frac{-c}{a_1}\right]^p \zeta[2p+1] \tag{3.55}$$

with $\gamma_E = 0.577216\ldots$. If $c/a_1 > 1$, then

$$\mathcal{J}_0[c;0,0] = \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon}+\ln\left[\pi\, c\, e^{\gamma_E-1}\right]\right] - \sum_{n_1\in\mathbf{Z}} \ln\left|1-e^{-2\pi\,\gamma(n_1)}\right|^2 + 4\sqrt{\frac{c}{a_2}} \sum_{\tilde{n}_1>0} \frac{K_1(s_{\tilde{n}_1})}{\tilde{n}_1}\,. \tag{3.56}$$

Here $\zeta[x]$ is the Riemann Zeta function; K_1 is the modified Bessel function, see appendix E for definitions. The pole structure is the same for both expressions of $\mathcal{J}_0$. Regarding the finite terms, $\mathcal{J}_0$ of eq. (3.55) has power-like terms in $c \sim k^2$ but these are suppressed by the radii/area of the compactification. These terms are the counterpart of the term[10] $c \ln c$ of eq. (3.56) in the case $c/a_1 \geq 1$. Note that in the first square bracket, $\mathcal{J}_0$ in (3.56) has a power-like dependence on $c \sim k^2$ whereas the last two terms in $\mathcal{J}_0$ are exponentially suppressed at large $c/a_1 \sim k^2 R_5^2$ and (given the symmetry $a_1 \leftrightarrow a_2$) also at large $c/a_2 \sim k^2 R_6^2$. The above expressions are important when we discuss the running of the effective gauge coupling and of the coupling of the higher derivative operator, after cancelling the divergence in eq. (3.53).

Let us consider some limiting cases. If $k^2 \ll \min(1/R_5^2, 1/R_6^2)$, eqs. (3.53), (3.55) give:

$$\Pi^{\rm hyper}(k,0) \approx \frac{i\sigma}{(4\pi)^2}\left\{\frac{\pi}{6} R_5 R_6 k^2 \left[\frac{-2}{\epsilon} - \ln\left[\pi e^{\gamma_E} \mu^2 R_5^2 \left|\eta(iR_6/R_5)\right|^{-4}\right]\right] -\ln\left[4\pi^2 e^{-2}\left|\eta(iR_6/R_5)\right|^4 R_6^2\, k^2\right]\right\} \tag{3.57}$$

where we used the Dedekind η function, see eq. (E.6). This result shows that after the addition of the higher derivative counterterm which will cancel the pole, the hypermultiplet only brings in a *logarithmic* dependence with respect to the momentum k^2, at values of k^2 much smaller than $1/R_{5,6}^2$. Note that this is a *low-energy* logarithm, originating from bulk contributions! If one evaluated instead $\Pi^{\rm hyper}(k^2=0,0)$, an IR mass regulator μ_{IR}^2 (replacing k^2) would still be required for mathematical consistency. This would then lead to a troublesome UV-IR mixing of type μ_{IR}^2/ϵ in (3.57), on which the limits $\mu_{IR} \to 0$ and $\epsilon \to 0$ do not commute. This would simply mean that the UV physics does not decouple in the low energy limit. This shows, even in the on-shell result for $\Pi^{\rm hyper}$, that there is a need for a higher derivative counterterm, for quantum consistency. We return to this issue in section 5.

In the case $k^2 \gg \max(1/R_5^2, 1/R_6^2)$, eqs. (3.53) and (3.56) give:

$$\Pi^{\rm hyper}(k,0) \approx \frac{i\sigma}{(4\pi)^2}\left\{\frac{\pi}{6} R_5 R_6 k^2 \left[\frac{-2}{\epsilon} - \ln\frac{\mu^2}{k^2} - \ln\left(4\pi e^{8/3-\gamma_E}\right)\right]\right\}. \tag{3.58}$$

[10] This term ($c \ln c$) will be important for the running of the higher derivative operator coupling, see later.

Finally, the brane correction $\Pi^{\rm local}$ of (3.52) also has a divergence. For any 6D momenta

$$\Pi^{\rm local}(k,\vec{k},\vec{k}')=\frac{i}{32\pi^2}\left\{\frac{2}{\epsilon}+\ln 4\pi\mu^2 e^{-\gamma_E}-\int_0^1 dx \ln\left[x(1-x)(k^2+\vec{k}'^2)+\left(\frac{\vec{k}}{2}+\left(x-\frac{1}{2}\right)\vec{k}'\right)^2\right]\right\} \tag{3.59}$$

which if $\vec{k}=\vec{k}'=0$ simplifies to:

$$\Pi^{\rm local}\big(k,0,0\big)=\frac{i}{32\pi^2}\left\{\frac{2}{\epsilon}+\ln 4\pi e^{2-\gamma_E}+\ln\frac{\mu^2}{k^2}\right\}, \tag{3.60}$$

where μ is the arbitrary (finite) scale introduced by the regularisation scheme.

The poles in $\Pi^{\rm hyper}$ and $\Pi^{\rm local}$ that we identified can be cancelled by introducing the following counterterms in the action:

$$\mathcal{L}_{c.t}=\int d^2z\, d^2\theta\left[\frac{1}{2h^2}{\rm Tr}\, W^\alpha \Box_6 W_\alpha+\frac{1}{2}\sum_{i=1}^{4}\frac{1}{g^2_{{\rm brane},i}}{\rm Tr}\, W^\alpha\, W_\alpha \delta^{(2)}(z-z_0^i)\right]+{\rm h.c.} \tag{3.61}$$

Here $z_0^i (i=1,\dots,4)$ are the fixed points of the $T^2/\mathbb{Z}_2$ orbifold considered. Further, h^2 is an additional dimensionless bulk coupling while $g_{{\rm brane},i}$ is a dimensionless brane coupling at the fixed point z_0^i. The introduction of such counterterms to cancel the poles is done up to an overall finite, unknown coefficient. As a result new parameters (couplings) emerge in the theory. For small compactification volume (or $k^2R_{5,6}^2\ll 1$), the bulk higher derivative operator is suppressed; however, for large radii (or $k^2R_{5,6}^2\gg 1$) it is relevant and important for the overall running of the zero-mode gauge coupling. The effect of this operator is largely ignored in the literature, both in effective field theory and string theory approaches. The renormalisation and the running of the coupling $h(k^2)$ will be considered in the next section.

Regarding the coupling $g_{{\rm brane},i}$, after its renormalisation there will be one additional parameter for the gauge kinetic term localised at each fixed point. If one considers such corrections in GUT models compactified on orbifolds [25], brane-localised gauge couplings respecting a gauge symmetry smaller than that in the bulk may be present. In that case the brane couplings are not universal and can affect the gauge coupling unification in such models [26].

4. "Running" of the effective gauge coupling as induced by the 6D theory

In this section we consider the one-loop renormalisation and running of the coefficients of the higher derivative operator and of the gauge kinetic term of the zero-mode gauge field.

To begin with, we consider the running of the bulk coupling h in (3.61) for the zero mode of the gauge field. After subtracting the divergence of the bulk term eq. (3.49) with eqs. (3.57) and (3.58) by a bulk higher derivative counterterm, one has the following momentum dependence of the renormalised h:

$$\begin{aligned}
k^2\ll\frac{1}{R_{5,6}^2}: \quad & \frac{4\pi}{h^2(k^2)}\approx\frac{4\pi}{h^2_{\rm tree}}+\Big[-C_2(G)+C(r)\Big]\frac{1}{96\pi^2}\ln\left[\pi e^{\gamma_E}\mu^2 R_5^2\big|\eta(iR_6/R_5)\big|^{-4}\right],\\
k^2\gg\frac{1}{R_{5,6}^2}: \quad & \frac{4\pi}{h^2(k^2)}\approx\frac{4\pi}{h^2_{\rm tree}}+\Big[-C_2(G)+C(r)\Big]\frac{1}{96\pi^2}\left\{\ln\frac{\mu^2}{k^2}+\ln 4\pi e^{8/3-\gamma_E}\right\}.
\end{aligned} \tag{4.1}$$

After writing each of these equations at two different momentum scales (for the same renormalisation scale μ) and subtracting them, we find that above the compactification scales the bulk coupling of the higher derivative operator runs logarithmically in k^2 while below the compactification scales it does not run. The running of $h(k^2)$ above the compactification scales is a just a bulk effect, little dependent on the details of localised singularities associated with the orbifold action[11]. Note that the higher derivative counterterm in eq. (3.61) "absorbed" all linear dependence on k^2 in eqs. (3.57) and (3.58), arising from eq. (3.55), (3.56), and this is relevant for the discussion below. For $k^2 R_{5,6}^2 \gg 1$ the coupling h is not suppressed, and this has implications for the running of the effective gauge coupling of the zero-mode gauge boson above the compactification scales.

Let us now investigate the running of the effective gauge coupling $g_{\rm eff}(k^2)$ which is defined as the coefficient of the gauge kinetic term of zero-mode gauge boson. The tree level value of the effective gauge coupling has contributions from both bulk and branes, including the bulk higher derivative term. It can be read off from the following gauge kinetic term:

$$-\frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}\left(\frac{1}{g_{\rm tree}^2}+\frac{1}{\sigma h_{\rm tree}^2}\Box_4\right)F^{\mu\nu}\right] \tag{4.2}$$

where

$$\frac{1}{g_{\rm tree}^2}\equiv\frac{1}{\sigma g^2}+\sum_{i=1}^{4}\frac{1}{g_{{\rm brane},i}^2},\quad \sigma\equiv\frac{1}{4\pi^2 R_5 R_6}. \tag{4.3}$$

Here g^2 and $g_{{\rm brane},i}^2$ are the tree-level gauge couplings in the bulk and at the fixed points, respectively. Note that, although the brane localised couplings $g_{{\rm brane},i}$ are new parameters introduced in the theory, the coupling $g_{\rm tree}$ only depends on their overall combination with the bulk gauge coupling g. Moreover, due to the new parameter $h_{\rm tree}$ of the higher derivative counterterm, ultimately, there is a momentum dependent contribution to the *effective* gauge coupling even at tree level.

After taking into account the radiative corrections (see (3.49), (3.50)) the zero-mode gauge coupling $g_{\rm eff}(k^2)$ is, at one-loop[12]:

$$\frac{1}{g_{\rm eff}^2(k^2)}=\frac{1}{g_{\rm tree}^2}-\frac{k^2}{\sigma h_{\rm tree}^2}+i\Big[C_2(G)-C(r)\Big]\frac{1}{\sigma}\,\Pi_*^{\rm hyper}(k,0)+4iC_2(G)\Pi_*^{\rm local}(k,0,0). \tag{4.4}$$

The subscript $*$ in the self-energy $\Pi_*^{\rm local}$ means that only the finite part of $\Pi^{\rm local}$ should be considered, because its singularity (the pole $2/\epsilon$) was cancelled by the tree level coupling $g_{\rm tree}$ in eq. (3.61). For the self-energy $\Pi_*^{\rm hyper}$ the subscript $*$ refers to the finite part of $\Pi^{\rm hyper}$ *after* the renormalisation of the coefficient of the higher derivative counterterm (4.1);

[11] See also the discussion in [8].

[12] Eq. (4.4) can be written in a form which separates massive from massless modes' contributions:

$$\frac{1}{g_{\rm eff}^2(k^2)}=\frac{1}{g_{\rm tree}^2}-\frac{k^2}{\sigma h_{\rm tree}^2}-i\Big[-C_2(G)+C(r)\Big]\frac{1}{\sigma}\,\Pi_{m,*}^{\rm hyper}(k,0)-i\Big[-3C_2(G)+C(r)\Big]2\,\Pi_*^{\rm local}(k,0,0)$$

where $\Pi_{m,*}^{\rm hyper}\equiv\Pi^{\rm hyper}-\Pi_{0,0}^{\rm hyper}$, with $\Pi_{0,0}^{\rm hyper}$ the (0,0) mode contribution and we used $\Pi_{0,0}^{\rm hyper}/\sigma=2\Pi^{\rm local}$. On this form we see the emergence of 4D $\mathcal{N}=2$ and $\mathcal{N}=1$ beta functions of massive and massless sectors.

therefore $\Pi_*^{\rm hyper}$ does not include the divergence k^2/ϵ in $\Pi^{\rm hyper}$ which corresponds to the renormalisation of $h_{\rm tree}$ in eq. (4.1). With these considerations, note that $g_{\rm tree}$ and $h_{\rm tree}$ in (4.4) and in the equations to follow denote only the finite part of tree level couplings.

Let us now address the running of $g_{\rm eff}(k^2)$ and the relation connecting it to the tree level coupling $g_{\rm tree}$. To begin with, consider first the case of $k^2 \ll 1/R_{5,6}^2$. To obtain the running of $g_{\rm eff}(k^2)$ for this region one writes (4.4) at two different momentum scales $q^2, k^2 \ll 1/R_{5,6}^2$ for the same renormalisation scale μ and subtracts them, then uses eqs. (3.57) and (3.60) to find:

$$\frac{4\pi}{g_{\rm eff}^2(q^2)} \approx \frac{4\pi}{g_{\rm eff}^2(k^2)} + \frac{1}{4\pi}\Big[-3C_2(G)+C(r)\Big]\ln\frac{k^2}{q^2}, \qquad \text{if} \quad q^2, k^2 \ll \frac{1}{R_{5,6}^2}. \tag{4.5}$$

This is an interesting result: we have obtained the familiar 4D logarithmic running of the effective gauge coupling with the usual 4D $\mathcal{N}=1$ beta function coefficient given by $b_1 = -3C_2(G)+C(r)$. Note that this running was derived from the full 6D theory, by taking into account both bulk and boundary loop effects. This is interesting because part of the above *logarithmic* running comes from the bulk[13], associated with the massless states. More explicitly, the logarithmic correction in (4.5) contains a "bulk" part $C(r)\ln k^2$ due to the hypermultiplet, while the vector multiplet provides a "bulk" part $-C_2(G)\ln k^2$ as well as a "brane" part $-2C_2(G)\ln k^2$, which added together give the beta function in (4.5). We note that the running of the effective coupling $g_{\rm eff}$ as shown in eq. (4.5) is unaffected by the higher derivative operators as long as we are in the region $k^2 \ll 1/R_{5,6}^2$.

The next step in our analysis is to establish a connection between the tree level coupling $g_{\rm tree}$ and the gauge coupling at low momentum scales well below the compactification scales ($k^2 \ll 1/R_{5,6}^2$), after integrating out all massive Kaluza-Klein modes[14]. Using again eq. (4.4) together with (3.57), (3.60), we have

$$\frac{4\pi}{g_{\rm eff}^2(k^2)} \approx \frac{4\pi}{g_{\rm tree}^2} - \frac{b_2}{4\pi}\ln\Big[4\pi e^{-\gamma_E}\big|\eta(i\,u)\big|^4\,u\,(4\pi^2 R_5 R_6\,\mu^2)\Big] - \kappa + \frac{b_1}{4\pi}\ln\frac{\xi_1\mu^2}{k^2}, \quad k^2 \ll \frac{1}{R_{5,6}^2},$$

$$\text{with} \quad \kappa \equiv 4\pi^2 k^2 R_5 R_6\left[\frac{4\pi}{h_{\rm tree}^2} + \frac{b_2}{96\pi^2}\ln\Big[\pi e^{\gamma_E}\mu^2 R_5 R_6 u^{-1}\big|\eta(iu)\big|^{-4}\Big]\right] \ll 1. \tag{4.6}$$

Here $u \equiv R_6/R_5$ and $\xi_1 = 4\pi e^{2-\gamma_E}$. Further $b_1 = -3C_2(G)+C(r)$ is the $\mathcal{N}=1$ beta function while $b_2 = -C_2(G)+C(r)$ is 1/2 of the $\mathcal{N}=2$ beta function coefficient on the torus, with 1/2 to account for the fact that the number of modes is reduced on $T^2/\mathbb{Z}_2$. As written, eq. (4.6) connects $g_{\rm eff}(k^2 \ll 1/R_{5,6}^2)$ to the tree level coupling $g_{\rm tree}$, *after* integrating out the massive Kaluza-Klein modes. The effect of these modes is accounted for by the term multiplied by b_2 in (4.6), as an overall threshold correction. It is important to note from (4.6) that the dominant contribution is of logarithmic dependence on k^2 and this is associated with the massless states only. Any power-like dependence of $g_{\rm eff}(k^2)$ on the

[13] See $\Pi^{\rm hyper}$ of (3.57).

[14] Early studies on this topic can be found in [35], but using instead an *on-shell* approach.

momentum scale is suppressed by the compactification volume, $\kappa \ll 1$, (i.e. the higher derivative operator is also suppressed.) This is the case after the renormalisation of the coupling h of the higher derivative gauge kinetic term, eqs. (3.61) and (4.1).

Eq. (4.6) can be used to study whether the low energy measurements of the couplings, e.g. electroweak scale values of the couplings are consistent with a common value $g_{\rm tree}$, regarded in this case as the "unified" coupling. The DR renormalisation scale μ is in this picture regarded as the unification scale. Eq. (4.6) is the counterpart of that computed in the (on-shell) string, in various models [17, 18, 16] (see also [35]). As we shall detail later, our result in (4.6) is more in agreement with that of the 4D Z_N orientifold models of type I strings [16], rather than that of the heterotic string [17, 18].

We have so far considered the behaviour of $g_{\rm eff}(k^2)$ at momentum scales $k^2 \ll 1/R_{5,6}^2$ and its relation to the tree level coupling. At higher momentum scales, the higher derivative operator becomes more important and one cannot neglect the presence of its coupling $h(k^2)$, eq. (4.1). The regime $k^2 \sim 1/R_{5,6}^2$ is that of dimensional crossover [15] and is the most difficult to investigate technically. In this case eqs. (3.57), (3.58) provide a rather poor approximation when used in eq. (4.4) to find $g_{\rm eff}$. One must use instead the full expressions of the functions $\mathcal{J}_0$, eqs. (3.55) and (3.56), integrated over x as in (3.53). These expressions converge even in the case $k^2 \sim 1/R_{5,6}^2$ and can be used to find the running of $g_{\rm eff}$ in this regime. These expressions are somewhat complicated and this prevents an intuitive, simple picture for this regime. In this case a full numerical approach based on (3.55), (3.56) may be more suitable.

Finally, let us consider the case of even higher momenta, $k^2 \gg 1/R_{5,6}^2$. In this case we find that the coupling $h(k^2)$ gives a substantial contribution to the running of the effective gauge coupling. From eq. (4.4) together with eqs. (3.58) and (3.60), we obtain the following result:

$$\frac{4\pi}{g_{\rm eff}^2(k^2)} \approx \frac{4\pi}{g_{\rm tree}^2} - 4\pi^2 k^2 R_5 R_6 \left[\frac{4\pi}{h_{\rm tree}^2} + \frac{b_2}{96\pi^2} \ln\frac{\mu^2\xi_2}{k^2}\right] - \frac{C_2(G)}{2\pi} \ln\frac{\mu^2\xi_1}{k^2}, \text{ if } k^2 \gg \frac{1}{R_{5,6}^2} \tag{4.7}$$

where $\xi_2 = 4\pi e^{8/3-\gamma_E}$, $\xi_1 = 4\pi e^{2-\gamma_E}$ are subtraction scheme dependent constants for the divergences of the bulk and brane contributions respectively[15]. The scale μ is the familiar renormalisation scale in the DR scheme, at which a "boundary" value of the coupling is provided.

Eq. (4.7) describes the running of the effective gauge coupling well above the compactification scales. The last term in eq. (4.7) is due to massless states (brane part only), which contribute to the running. Further, the square bracket accounts for the contribution coming from the running coefficient of the higher derivative term. Since the square bracket involves $k^2 R_5 R_6$ which essentially counts the number of excited Kaluza-Klein modes, we obtain a power-like running with respect to the *momentum* scale, valid above the compactification scales. Note, however, that the power dependence on k^2 is controlled by the parameter $h_{\rm tree}^2$ which multiplies it (and is also affected by the presence of $\ln\xi_2$ which is a

[15]Remember that these are in the minimal subtraction scheme, i.e. only the poles in ϵ were cancelled by $g_{\rm tree}$ and $h_{\rm tree}$.

subtraction scheme dependent coefficient). We therefore need a deeper understanding of this coefficient.

To this purpose, let us address the origin of the power-like term and explain what ultimately controls it. To do so we rewrite eq. (4.4) as

$$\frac{4\pi}{g^2_{\rm eff}(k^2)} = \frac{4\pi}{g^2_{\rm tree}} - \frac{4\pi}{h^2(k^2)}\,(4\pi^2 k^2 R_5 R_6) + \frac{b_2}{4\pi}\,\delta - \frac{C_2(G)}{2\pi}\ln\frac{\mu^2\xi_1}{k^2}. \tag{4.8}$$

This equation is valid at all values of k^2, large or small relative to $1/R^2_{5,6}$, provided that other higher dimension operators are negligible. Here δ is the integral over x as in (3.53) of the part in $\mathcal{J}_0$ of either (3.55) or (3.56) which does not contain the first square bracket in these two equations. If $k^2 \ll 1/R^2_{5,6}$ then δ gives a log running given by the last term in (3.57) while if $k^2 \gg 1/R^2_{5,6}$ then $\delta \approx 0$. With these values of δ and with the running of $h(k^2)$ as in (4.1) one recovers the limiting cases of large and small momenta discussed in (4.6) and (4.7).

The interpretation of the result in (4.8) is as follows: the coefficient of the power-like term $k^2\,R_5R_6$ is ultimately controlled by the renormalised coupling $h(k^2)$ of the higher derivative term in the action and by its running. In some works the notion "power running" refers to power-like (threshold) corrections in the UV cutoff *regulator* as opposed to the power-like dependence with respect to the momentum scale that we obtained here, and these are not to be confused. Our result above clarifies that the power running with respect to the momentum scale is controlled by the one-loop corrected coupling of the higher derivative gauge kinetic term in the action.

In general, in theories with higher derivative operators additional effects are present. One should essentially start with the full action *including* at the tree level the higher derivative gauge kinetic term, and quantise the theory in its presence. This is a rather difficult problem. Further, in the presence of the higher derivative operator, the propagator of the zero-mode gauge boson changes into a sum of two terms: one particle-like propagator and one ghost-like propagator, respectively[16]:

$$G(k) = \frac{-ig_{\mu\nu}}{k^2\left(\frac{k^2}{h^2}+\frac{1}{g^2}\right)} = -ig^2 g_{\mu\nu}\left[\frac{1}{k^2} - \frac{1}{k^2+\frac{h^2}{g^2}}\right]. \tag{4.9}$$

From the coefficient of each term, one can see that both particle and ghost have the same coupling g to matter fields. Although the ghost pole is located around the 6D fundamental scale, the ghost state may give an additional non-vanishing threshold correction to the gauge coupling. Further, there are many other complications, specific to higher derivative theories, such as unitarity violation, non-locality, etc, see [27]–[33], which made the study of these theories less popular. Another difficulty that arises is that one must also take into account the effect of brane-localised terms on the spectrum of the Kaluza-Klein modes [34], not considered in this paper. Therefore, a detailed investigation of models with higher derivative operators is far more complicated and beyond the purpose of the present work.

[16]See section 4 in [6] for a similar discussion for the case of a massive scalar field.

To conclude, the higher derivative operator must be included to ensure the quantum consistency of the model with extra dimensions, and therefore plays an important role in the running of the effective gauge coupling. After the renormalisation of its coupling h there is only a logarithmic dependence on the momentum scale of the 4D effective gauge coupling $g_{\rm eff}(k^2 \ll 1/R_{5,6}^2)$. At a higher momentum scale power-like terms in $k^2 R_5 R_6 < 1$ are present. At even higher momentum scales $k^2 \gg 1/R_{5,6}^2$, the higher derivative operator is important and its coupling $h(k^2)$ has a logarithmic running with respect to k^2. In this case the effective gauge coupling has, after renormalisation of h, a power-like dependence on the momentum scale. The coefficient of this power-like term in momentum is equal to the running coupling of the higher derivative operator. These findings provide a clear explanation of the power-like running (with respect to the momentum scale) of the gauge couplings in models with extra dimensions.

5. Higher derivative operators in other schemes and in string theory

It is interesting to investigate how higher derivative counterterms emerge in other regularisation schemes and in string theory as well. This is important because their role in ensuring the quantum consistency of the models was largely ignored in the literature. To this purpose, we consider the effects of the massive Kaluza-Klein modes in a regularisation with a momentum cutoff, i.e. the proper-time cutoff regularisation. Note that a proper-time cutoff is less suitable as a regulator, since it breaks 4D Lorentz invariance and Ward identities. Nevertheless, its use provides a more intuitive picture and will help our physical understanding of the important role of higher derivative operators.

Let us introduce a cutoff regulator $1/\Lambda^2$ in $\Pi^{\rm hyper}$ of (3.53) and consider this equation for the *massive* mode contributions only, denoted $\Pi_m^{\rm hyper}$, i.e. we exclude the $(0,0)$ mode[17]. One has

$$\begin{aligned}\Pi_m^{\rm hyper}(k^2,0) &= \frac{i\pi^2\sigma}{(2\pi)^4}\int_0^1 dx \sum_{n_{1,2}\in\mathbf{Z}}{}' \int_{1/\Lambda^2}^{\infty}\frac{dt}{t}\, e^{-\pi\, t\,[k^2 x(1-x)+n_1^2/R_5^2+n_2^2/R_6^2]} \\ &= \frac{i\sigma}{(4\pi)^2}\Big\{\Lambda^2 R_5 R_6 - \ln\Big[4\pi e^{-\gamma_E}\,|\eta(i\,u)|^4\,u\,\big(\Lambda^2 R_5 R_6\big)\Big] \\ &\quad -\frac{\pi}{6}k^2 R_5 R_6 \ln\Big[(4\pi)^{-1}e^{\gamma_E}\Lambda^2 R_5 R_6 u^{-1}\big|\eta(iu)\big|^{-4}\Big]\Big\}\end{aligned} \tag{5.1}$$

which is valid only if $k^2 \ll 1/R_{5,6}^2 \ll \Lambda^2$. The prime on the double sum marks the absence of the $(0,0)$ mode. The $\ln\Lambda$ term in the square bracket is the counterpart of the $-2/\epsilon$ pole in

[17]The $(0,0)$ mode combines with the contribution of $\Pi^{\rm local}$ to give 4D $\mathcal{N}=1$ beta function, see footnote [12]

the DR scheme[18], first term in (5.2). The $k^2 \ln\Lambda$ term corresponds the k^2/ϵ term in the DR scheme, associated with higher derivative operator. These divergences are cancelled by the bulk kinetic term and the higher derivative operator, respectively. In addition we obtain a quadratic divergence in the *regulator* Λ (5.1) which cannot appear in the DR scheme.

To see in more detail the need for a higher derivative operator in this regularisation, remember that the momentum k^2 may be regarded as an IR regulator, to ensure the finiteness (at $t\rightarrow\infty$) of $\Pi^{\rm hyper}$ in (5.1) when the massless mode $(n_1,n_2)=(0,0)$ is included. One notices that in the last term of (5.1) the limits $k^2\rightarrow 0$ and $\Lambda^2\rightarrow\infty$ do not commute [14]:

$$\Big[k^2\rightarrow 0, \Lambda^2\rightarrow\infty\Big]\neq 0. \tag{5.3}$$

We therefore have a rather troublesome UV-IR mixing term (UV divergent, IR finite) meaning that the two sectors of the theory are not decoupled at the quantum level! As we recall from the comment following (3.57), a similar UV-IR mixing in the DR scheme was cancelled by the renormalisation of a higher derivative counterterm. In a similar way, the renormalisation of this operator cancels the log divergence in the last term of (5.1) so that it enables the decoupling of the IR from the UV regime. Finally, the logarithmic and quadratic divergences in the first two terms of (5.1) have to be subtracted by the gauge kinetic counterterm at a renormalisation point. However, there remains a correction $\Lambda^2 R_5 R_6$ with arbitrary coefficient[19], which may eventually be identified from a more fundamental theory, e.g. from the field theory limit of the heterotic string [14, 36].

What does string theory say about these problems or about the need for higher derivative operators at the quantum level? To begin with, it is interesting to observe that in 4D Z_N orientifold models of type I strings [16], the one loop threshold corrections associated with the massive $\mathcal{N}=2$ sector are exactly of the type in (4.6) *after* the tadpole cancellation condition. Note that this condition "removes" any power-like dependence on the string scale. This similarity of the results is interesting, although there does not seem to exist a clear field theoretic understanding of this tadpole cancellation condition and what that means for the higher derivative operator that we found. This also raises intriguing issues such as whether the higher derivative counterterm that emerged and is relevant at large radii may be related to the non-perturbative effects of D-branes.

Next, let us consider the case of the heterotic string toroidal orbifolds T^6/Z_N, N even, with "fixed" two-torus under the orbifold action. This brings one-loop string threshold

[18]In the DR scheme, the massive sector (this excludes the (0,0) mode) gives for $k^2\ll 1/R_{5,6}^2$ (eq. (3.57))

$$\begin{aligned}\Pi_m^{\rm hyper}(k^2,0) &= \frac{i\pi^2\sigma\mu^\epsilon}{(2\pi)^{4-\epsilon}}\int_0^1 dx\sum_{n_{1,2}\in\mathbf{Z}}{}^{'}\int_0^\infty\frac{dt}{t^{1-\epsilon/2}}\,e^{-\pi\, t\,[k^2x(1-x)+n_1^2/R_5^2+n_2^2/R_6^2]}\\ &= \frac{i\sigma}{(4\pi)^2}\bigg\{\frac{-2}{\epsilon}-\ln\Big[4\pi e^{-\gamma_E}\,|\eta(i\,u)|^4\,u\,\big(4\pi^2\mu^2R_5R_6\big)\Big]\\ &\quad+\frac{\pi}{6}k^2R_5R_6\bigg[\frac{-2}{\epsilon}-\ln\Big[\pi e^{\gamma_E}\mu^2R_5R_6u^{-1}\big|\eta(iR_6/R_5)\big|^{-4}\Big]\bigg]\bigg\}.\end{aligned} \tag{5.2}$$

[19]One must not forget that Λ is actually a regulator and $100\times\Lambda$ is equally good a choice!

corrections due to the $\mathcal{N}=2$ massive sector of Kaluza-Klein and winding modes [17, 18]. In the limit of large radii (in units α') non-perturbative effects (world-sheet instanton effects) are suppressed to give in the field theory regime:

$$\Pi^{\rm hyper}(k^2=0,0) \sim -\ln\left[4\pi e^{-\gamma_E}|\eta(iu)|^4\, u\, T_2\right] + \frac{\pi}{3}T_2 + \epsilon_{IR}\ln\alpha', \tag{5.4}$$

where $T_2 = R_5R_6/\alpha'$; u is the usual complex structure (assuming an orthogonal fixed two-torus). This result is similar to that in (5.1) for $k^2=0$, as discussed in detail in [14, 36].

Although the string provides only an *on-shell* result ($k^2=0$), the one-loop string nevertheless requires an infrared regulator denoted ϵ_{IR}, which plays a role similar to a small momentum $k^2 \to 0$. The last term in (5.4) vanishes when the infrared regulator in string is removed $\epsilon_{IR}\to 0$, assuming α' non-zero. However, $\alpha'^{-1} \sim M^2_{\rm string}$ is the string scale, which is the counterpart to our UV momentum cutoff regulator Λ^2 [36, 14]. One immediately observes from the last term in (5.4) that the limit of removing the infrared regulator $\epsilon_{IR}\to 0$ and the limit of large $M_{\rm string}$ or $\alpha'\to 0$ which is the effective field theory regime, do not commute:

$$\left[\epsilon_{IR}\to 0, \alpha'\to 0\right] \neq 0. \tag{5.5}$$

This is the same problem we encountered in the proper-time cutoff regularisation scheme, if we regard ϵ_{IR} as $k^2\to 0$ and $M_{\rm string}\to\infty$ as the counterpart of Λ^2. Therefore there is again a UV-IR mixing and a non-decoupling of the high scale physics i.e. of massive modes from the 4D low energy limit [14], also encountered in the DR scheme (see comment after (3.57)). The reason why such effects are usually not discussed in string theory is ultimately related to the underlying *on-shell* approach, which "obscures" the need for higher derivative counterterms. The last term in (5.4) is then a "remnant" of such effects, and a reminder of this issue in the heterotic string. This non-decoupling of massive modes in the low-energy (4D) raises questions on the consistency of attempts to match string unification scale (in the presence of such thresholds) with MSSM-like unification scenarios. This underlines the need for a study of the higher derivative operators in string theory[20].

6. Conclusions

In this paper we performed a general analysis of the one-loop corrections to the self-energy of gauge bosons in the framework of 6D $\mathcal{N}=1$ supersymmetric gauge theories on orbifolds. We first considered an Abelian gauge theory using the Feynman diagram approach in the component field formalism. The analysis was then extended to the case of non-Abelian gauge theories on orbifolds. By employing the background field method in higher dimensions, we established the general setup for the one-loop effective action for gauge bosons and then applied it to the case of the orbifold $T^2/\mathbb{Z}_2$. As a consequence, we have shown that our component field approach is consistent with and complementary to the superfield calculation [9, 10]. Moreover, the additional benefit of our component field approach is that our findings can be easily used in a non-supersymmetric setup.

[20] For more details on this matter see [14] and section 3 in [11].

In the case of Abelian theories on $T^2/\mathbb{Z}_2$ we computed the divergent and finite parts of the one-loop correction to the vacuum polarisation tensor. For the case of a bulk fermion it was shown that only bulk corrections are present. The bulk corrections contained a divergence which had to be cancelled by the introduction of a 6D higher derivative counterterm. The loop corrections of a bulk scalar to the gauge boson self-energy were also computed to show that there is a bulk (6D) higher derivative as well as brane localised (4D) gauge kinetic counterterms. The former is absent in the limit when the two compact dimensions collapse onto each other (similar for the bulk fermion), in agreement with the result that there is no higher derivative counterterm from the gauge interactions at one loop in 5D[21]. Combining the bulk scalar and fermion contributions, we showed that a hypermultiplet only gives a bulk correction which requires a higher derivative counterterm, in agreement with other recent studies [10].

The above one-loop results were generalised to the case of non-Abelian gauge theories on the $T^2/\mathbb{Z}_2$ orbifold and many of our results are expected to apply to other 6D orbifolds as well. This generalisation was done by first constructing the effective action with a background field method in higher dimensions, which was then applied to 6D orbifolds. To this purpose, we introduced functional differentiations compatible with the orbifold actions on the fields. We found that hypermultiplets provide only bulk corrections, while vector multiplets bring in both bulk and boundary-localised corrections. The divergence of the bulk correction is cancelled by a 6D higher derivative counterterm while the divergence of the brane correction requires 4D boundary-localised gauge kinetic counterterms. Therefore, after subtraction of divergences, there are unknown new parameters (couplings) coming from these operators in the theory. The bulk correction has a non-perturbative origin since we re-summed infinitely many individual (divergent) loop contributions of the bulk modes. At the technical level this is related, in part, to a singularity (simple pole) of the Hurwitz-Riemann Zeta function in the re-summed correction. We also computed the finite part of the bulk correction which gives the momentum dependence of the self energy of the gauge boson. After renormalisation of the higher derivative operator, the finite part of the bulk correction has, at $k^2 \ll 1/R_{5,6}^2$, a familiar, logarithmic dependence on k^2 due to the massless states only. There are in addition power-like terms (in $k^2 R_5 R_6 \ll 1$), strongly suppressed in this regime, and due to integrated massive modes. At higher scales the finite part contains power-like and exponentially suppressed terms in $k^2 R_5 R_6$.

We then studied the behaviour of the effective 4D gauge coupling $g_{\rm eff}(k^2)$, which was defined as the coupling of the zero-mode gauge boson. After renormalisation of the higher derivative operator coupling, we discussed in detail the running of the effective gauge coupling with respect to the momentum scale. In the limit of momenta much smaller than the compactification scales, the effective coupling runs logarithmically with the 4D $\mathcal{N} = 1$ beta function and this low-scale running is induced by both bulk and brane terms.

We also analysed in detail the threshold corrections to the low energy gauge couplings, due to massive Kaluza Klein modes with $\mathcal{N} = 2$ beta function coefficient. The relation of the low energy effective coupling to the tree level coupling shows that there is only

[21]Localised superpotential interactions do bring in one-loop higher derivative counterterms in 5D [5, 6].

a *logarithmic* dependence of $g_{\rm eff}(k^2)$ on the momentum scale, while power-like terms are strongly suppressed in the regime $k^2 R_5 R_6 \ll 1$. This finding has potentially interesting consequences for phenomenology, such as the unification of the gauge couplings. This is the result after the renormalisation of the higher derivative coupling, which below compactification scale is essentially constant (no running). It was observed that this result was in agreement with that of the 4D Z_N orientifolds of the type I string, where no power-like terms are present in the one-loop threshold correction to the low-energy coupling.

At higher momentum scales, the higher derivative gauge kinetic term is more important. After renormalisation, its coupling has a logarithmic running with respect to the momentum scale. At $k^2 \sim 1/R_{5,6}^2$ we provided technical formulae which allow the study of the dimensional cross-over regime of the effective gauge coupling. At larger momentum scales ($k^2 \geq 1/R_{5,6}^2$), the initially negligible contribution of the higher derivative term to the coupling $g_{\rm eff}$ becomes significant and starts to change the running of the effective coupling with respect to momentum scale from the logarithmic one to the power-like one. This behaviour was studied in detail. At all momentum scales the coefficient of the power-like term is equal to the running coupling of the higher derivative gauge kinetic term. This is an interesting finding which clarifies the physical meaning of power-like running (in momentum) in models with extra dimensions.

Finally, the importance of the higher derivative operator was emphasised by showing the need for them as counterterms in other regularisation schemes and in (heterotic) string theory. In particular, it was shown that in these cases there is a UV-IR mixing (UV divergent, IR finite) at the quantum level, due to ignoring the quantum role of the higher derivative operator. In the (on-shell) heterotic string this can be seen from the fact that the field theory limit of the one-loop correction from massive states does not commute with the infrared regularisation of the one-loop string. This underlines the need for the investigation of the role of higher derivative operators in string theory too.

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A. Notations and conventions

The metric has the signature $g_{MN} = \mathrm{diag}(+ - - - - -)$; $M, N = 0, 1, 2, 3, 5, 6$ are six-dimensional indices and $\mu, \nu = 0, 1, 2, 3$ are four-dimensional ones. The Clifford algebra in six dimensions is characterised by

$$\{\Gamma^M, \Gamma^N\} = 2g^{MN}, \quad (\Gamma^M)^T = -C\Gamma^M C^{-1}, \quad C^T = C, \quad C^\dagger = C^{-1}. \tag{A.1}$$

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An explicit representation for the 8×8 gamma-matrices is

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} 0 & \gamma^5 \\ \gamma^5 & 0 \end{pmatrix}, \quad \Gamma^6 = \begin{pmatrix} 0 & -\mathbb{1}_4 \\ \mathbb{1}_4 & 0 \end{pmatrix} \tag{A.2}$$

where γ^μ and γ^5 are the four-dimensional gamma matrices, with

$$\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3 = -i \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \tag{A.3}$$

In this basis, the six-dimensional chirality operator is diagonal:

$$\Gamma^7 = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^5\Gamma^6 = \begin{pmatrix} -\mathbb{1}_4 & 0 \\ 0 & \mathbb{1}_4 \end{pmatrix}. \tag{A.4}$$

The charge conjugation is then

$$C = \begin{pmatrix} 0 & -C_5 \\ C_5 & 0 \end{pmatrix} \tag{A.5}$$

where C_5 is the five-dimensional charge conjugation.

After imposing the chirality constraint in six dimensions, the gamma matrices acting on right-handed or left-handed 6D spinors are reduced to the following 4×4 matrices, respectively,

$$\gamma^M \equiv (\gamma^\mu, \gamma^5, -\mathbb{1}_4) \quad \text{and} \quad \bar{\gamma}^M \equiv (\gamma^\mu, \gamma^5, \mathbb{1}_4). \tag{A.6}$$

In five dimensions, the gamma matrices $\Gamma^a (a = 0, 1, 2, 3, 5)$ are given by

$$\Gamma^\mu = \gamma^\mu, \quad \Gamma^5 = \gamma^5 \tag{A.7}$$

satisfying the following relations:

$$(\Gamma^a)^T = -C_5\Gamma^a C_5^{-1}, \quad C_5^T = -C_5, \quad C_5^\dagger = C_5^{-1}. \tag{A.8}$$

We note some useful formulae for the traces, used in the text

$$\begin{aligned} \mathrm{Tr}[\gamma_\mu\gamma_\nu] &= 4g_{\mu\nu}, \\ \mathrm{Tr}[\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma] &= 4(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\rho\nu}), \\ \mathrm{Tr}[\gamma_\mu\gamma_\rho\gamma_5\gamma_\nu\gamma_\sigma] &= -4i\epsilon_{\mu\rho\nu\sigma}, \\ \mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\sigma] &= \mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_5] = \mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\nu\gamma_5] = 0. \end{aligned} \tag{A.9}$$

In the text we also used the following relations on Casimir operators for a representation r (denoted G (N) in the case of the adjoint (fundamental) representation) of the group $\mathcal{G}$:

$$\mathrm{tr}(t_G^a t_G^b) = C_2(G)\delta_{ab}, \quad \mathrm{tr}(t_r^a t_r^b) = C(r)\delta^{ab}. \tag{A.10}$$

with $C_2(G) = C(G) = N$, $C(N) = 1/2$ and $C_2(N) = (N^2 - 1)/2N$, in the case of SU(N).

B. Propagators of bulk fields on orbifolds

We present in the following the propagators on the $T^2/\mathbb{Z}_2$ orbifold used in the text. On the orbifold $T^2/\mathbb{Z}_2$, the positions $z \equiv (x_5, x_6)$ in the extra dimensions are identified by $z \to -z$. For a bulk fermion, we impose the boundary conditions as

$$\begin{aligned} P\psi(x,z) &\equiv i\eta_f\gamma_5\psi(x,-z) = \psi(x,z), \\ \psi(x,z) &= \psi(x,z+2\pi R_5) = \psi(x,z+i2\pi R_6) \end{aligned} \tag{B.1}$$

with $\eta_f = \pm 1$. Then, the fermion on the orbifold is written in terms of a fermion on T^2 as

$$\begin{aligned} \psi(x,z) &= \frac{1}{2}(1+P)\chi(x,z) \\ &= \frac{1}{2}(\chi(x,z) + i\eta_f\gamma_5\chi(x,-z)). \end{aligned} \tag{B.2}$$

By using the fermion propagator on T^2 given by

$$D(x,z;x',z') \equiv \langle \chi(x,z)\overline{\chi}(x',z')\rangle \to \tilde{D}(p,\vec{p},\vec{p}') \equiv \frac{i\delta_{\vec{p},\vec{p}'}}{\not{p} + \gamma_5 p_5 + p_6}, \tag{B.3}$$

we find the fermion propagator on the $T^2/\mathbb{Z}_2$ orbifold as

$$\begin{aligned} &D_{\eta_f}(x,z;x',z') \equiv \langle \psi(x,z)\overline{\psi}(x',z')\rangle \\ &\to \tilde{D}_{\eta_f}(p,\vec{p},\vec{p}') \equiv \frac{i}{2}\left(\frac{\delta_{\vec{p},\vec{p}'}}{\not{p} + \gamma_5 p_5 \pm p_6} - \eta_f \frac{\delta_{\vec{p},-\vec{p}'}}{\not{p} + \gamma_5 p_5 \pm p_6} i\gamma_5\right). \end{aligned} \tag{B.4}$$

Here $\pm$ depends on the $6D$ chirality. Now we consider a bulk scalar field satisfying the boundary conditions on the orbifold as

$$\begin{aligned} P\phi(x,z) &\equiv \eta_s\phi(x,-z) = \phi(x,z), \\ \phi(x,z) &= \phi(x,z+2\pi R_5) = \phi(x,z+i2\pi R_6) \end{aligned} \tag{B.5}$$

with $\eta_s = \pm 1$. Similarly to the fermion case, we can write down the scalar on the orbifold in terms of a scalar on the covering space as

$$\begin{aligned} \phi(x,z) &= \frac{1}{2}(1+P)\varphi(x,z) \\ &= \frac{1}{2}(\varphi(x,z) + \eta_s\varphi(x,-z)). \end{aligned} \tag{B.6}$$

Then, we obtain the scalar field propagator on the orbifold as

$$G_{\eta_s}(x,z;x',z') \equiv \langle \phi(x,z)\overline{\phi}(x',z')\rangle \to \tilde{G}_{\eta_s}(p,\vec{p},\vec{p}') \equiv \frac{i}{2}\frac{\delta_{\vec{p},\vec{p}'} + \eta_s\delta_{\vec{p},-\vec{p}'}}{p^2 - p_5^2 - p_6^2}. \tag{B.7}$$

C. Details of the one-loop vacuum polarisation to U(1) gauge bosons

We discuss in the following the detailed derivation of the one-loop vacuum polarisation of U(1) gauge bosons due to the fermionic and bosonic contributions.

C.1 A bulk fermion contribution

After introducing a Feynman parameter and shifting the integration momentum, we obtain the fermionic correction (2.9) as

$$\Pi^f_{\mu\nu} = -2g^2\delta_{\vec{k},\vec{k}'}\mu^{4-d}\sum_{\vec{p}'}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-\Delta)^2}\Big\{2p_\mu p_\nu - 2x(1-x)k_\mu k_\nu$$

$$+g_{\mu\nu}[-p^2+x(1-x)k^2+\vec{p}'\cdot(\vec{p}'+\vec{k}')]\Big\} \tag{C.1}$$

with

$$\Delta \equiv -x(1-x)(k^2-\vec{k}'^2)+(\vec{p}'+x\vec{k}')^2. \tag{C.2}$$

After re-writing the terms proportional to $g_{\mu\nu}$ as

$$-p^2+x(1-x)k^2+\vec{p}'\cdot(\vec{p}'+\vec{k}') = -(p^2-\Delta)+2x(1-x)(k^2-\vec{k}'^2)$$

$$+(1-2x)\vec{k}'\cdot(\vec{p}'+x\vec{k}'), \tag{C.3}$$

the correction becomes

$$\Pi^f_{\mu\nu} = -2g^2\delta_{\vec{k},\vec{k}'}\mu^{4-d}\sum_{\vec{p}'}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\Bigg\{\left[\frac{2p_\mu p_\nu}{(p^2-\Delta)^2}-\frac{g_{\mu\nu}}{p^2-\Delta}\right]$$

$$+\frac{1}{(p^2-\Delta)^2}\Big[2x(1-x)[(k^2-\vec{k}'^2)g_{\mu\nu}-k_\mu k_\nu]+(1-2x)\vec{k}'\cdot(\vec{p}'+x\vec{k}')g_{\mu\nu}\Big]\Bigg\}. \tag{C.4}$$

By using

$$\int\frac{d^dp}{(2\pi)^d}\left[\frac{2p_\mu p_\nu}{(p^2-\Delta)^2}-\frac{g_{\mu\nu}}{p^2-\Delta}\right]=0,$$

we end up with the result

$$\Pi^f_{\mu\nu} = -2g^2\delta_{\vec{k},\vec{k}'}\mu^{4-d}\sum_{\vec{p}'}\int_0^1 dx\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-\Delta)^2}$$

$$\times\Big(2x(1-x)[(k^2-\vec{k}'^2)g_{\mu\nu}-k_\mu k_\nu]+(1-2x)\vec{k}'\cdot(\vec{p}'+x\vec{k}')g_{\mu\nu}\Big). \tag{C.5}$$

used in the text, eq. (2.9).

C.2 A bulk scalar contribution

After using a Feynman parameter and a shift of integration momentum, the bosonic bulk contribution (2.17) is given by

$$\Pi^{\rm bulk}_{\mu\nu} \equiv -\frac{1}{2}g^2\delta_{\vec{k},\vec{k}'}\mu^{4-d}\sum_{\vec{p}'}\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-\Delta)^2}\Big\{-4p_\mu p_\nu-(1-2x)^2k_\mu k_\nu$$

$$+2g_{\mu\nu}[p^2+(1-x)^2k^2-(\vec{p}'+\vec{k}')^2]\Big\}. \tag{C.6}$$

Rewriting the terms proportional to $g_{\mu\nu}$ as

$$p^2 + (1-x)^2 k^2 - (\vec{p}' + \vec{k}')^2 = (p^2 - \Delta) + (1 - 3x + 2x^2)(k^2 - \vec{k}'^2) + 2(x-1)\vec{k}'(\vec{p}' + x\vec{k}'), \tag{C.7}$$

the bulk correction becomes

$$\begin{aligned}\Pi^{\rm bulk}_{\mu\nu} = & -\frac{1}{2} g^2 \delta_{\vec{k},\vec{k}'} \mu^{4-d} \sum_{\vec{p}'} \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \bigg\{ -2 \left[\frac{2p_\mu p_\nu}{(p^2-\Delta)^2} - \frac{g_{\mu\nu}}{p^2 - \Delta} \right] \\ & + \frac{1}{(p^2-\Delta)^2} \bigg[2(1-3x+2x^2)(k^2 - \vec{k}'^2) g_{\mu\nu} - (1-2x)^2 k_\mu k_\nu \\ & + 4(x-1)\vec{k}' \cdot (\vec{p}' + x\vec{k}') g_{\mu\nu} \bigg] \bigg\}. \end{aligned} \tag{C.8}$$

Then, after 4D momentum integration with eq. (C.5), the first two terms cancel. Now observe that

$$\frac{(1-2x)(k^2 - \vec{k}'^2)}{(p^2-\Delta)^2} = -\frac{\partial}{\partial x}\left(\frac{1}{p^2 - \Delta}\right) + \frac{2\vec{k}' \cdot (\vec{p}' + x\vec{k}')}{(p^2-\Delta)^2}.$$

Then from the x-integration

$$\int_0^1 dx \frac{\partial}{\partial x}\left(\frac{1}{p^2-\Delta}\right) = \frac{1}{p^2 - (\vec{p}' + \vec{k}')^2} - \frac{1}{p^2 - \vec{p}'^2},$$

we note that the surface term for the Feynman parameter vanishes after the Kaluza-Klein summation with the discrete shift in $\vec{p}'$. Therefore, we obtain the correction as

$$\begin{aligned}\Pi^{\rm bulk}_{\mu\nu} = & -\frac{1}{2} g^2 \delta_{\vec{k},\vec{k}'} \mu^{4-d} \sum_{\vec{p}'} \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2-\Delta)^2} \\ & \times \Big((1-2x)^2 [(k^2 - \vec{k}'^2) g_{\mu\nu} - k_\mu k_\nu] + 2(2x-1)\vec{k}'(\vec{p}' + x\vec{k}') g_{\mu\nu} \Big). \end{aligned} \tag{C.9}$$

used in the text, eq. (2.18).

D. Results and evaluation of series $\mathcal{J}_{0,1}$ for 6D orbifolds

We evaluate (with $c \geq 0$, $a_{1,2} > 0$, $0 \leq c_{1,2} < 1$):

$$\begin{aligned}\mathcal{J}_v[c; c_1, c_2] \equiv & \; \Gamma[\epsilon/2] \sum_{n_1, n_2 \in \mathbf{Z}} (n_1 + c_1)^v \Big[\pi [c + a_1(n_1+c_1)^2 + a_2(n_2+c_2)^2] \Big]^{-\epsilon/2} \\ = & \sum_{n_1, n_2 \in \mathbf{Z}} (n_1 + c_1)^v \int_0^\infty \frac{dt}{t^{1-\epsilon/2}} \, e^{-\pi t [c + a_1(n_1+c_1)^2 + a_2(n_2+c_2)^2]}, \quad v = 0, 1 \ldots; \end{aligned} \tag{D.1}$$

This expression was used in the text for $v = 0$ and $v = 1$ in eqs. (2.10), (2.11), (2.12), (2.22), (2.23), (3.51). In these eqs we assumed $a_i = 1/R^2_{i+4}$, $i = 1, 2$, $c_1 = x R_5 k'_5$, $c_2 = x R_6 k'_6$ and

$c=x(1-x)(k^2+\vec{k}'^2)$ in Euclidean metric. Since we can always shift c_i by an integer, only their fractional part will enter the final result.

The final value of $\mathcal{J}_0$ was given in [6] but in the text we also need to evaluate $\mathcal{J}_1$ however. Since the proof is similar, and to be general, we present the generic steps to evaluate $\mathcal{J}_v$. The counterpart of $\mathcal{J}_v$ with a factor $(n_2+c_2)^v$ in front of the integral is obtained from the replacements $c_1 \leftrightarrow c_2$ and $a_1 \leftrightarrow a_2$. Most important for us is to identify the poles of $\mathcal{J}_v$, (to find the counterterms) but we also evaluate the finite part which require us compute the $\mathcal{O}(\epsilon)$ term in the double sum in the first line in (D.1). Notation used:

$$\gamma(n_1) \equiv \frac{\sqrt{z(n_1)}}{\sqrt{a_2}} - i\, c_2; \qquad z(n_1) \equiv c + a_1(n_1+c_1)^2, \qquad u \equiv \sqrt{a_1/a_2} \tag{D.2}$$

Keeping the sum over n_1 fixed, we re-sum (see (E.4)) over n_2, so that

$$\begin{aligned}\sum_{n_{1,2}\in\mathbf{Z}} e^{-\pi t\,[a_2(n_2+c_2)^2+a_1(n_1+c_1)^2]} &= \sum_{n_2\in\mathbf{Z}} e^{-\pi\, t\,[a_2(n_2+c_2)^2+a_1 c_1^2]} + {\sum_{n_1\in\mathbf{Z}}}' \sum_{n_2\in\mathbf{Z}} e^{-\pi\, t\,[a_2(n_2+c_2)^2+a_1(n_1+c_1)^2]} \\ &= \sum_{n_2\in\mathbf{Z}} e^{-\pi\, t\,[a_2(n_2+c_2)^2+a_1 c_1^2]} + \frac{1}{\sqrt{t\, a_2}} {\sum_{n_1\in\mathbf{Z}}}' e^{-\pi t\, a_1\,(n_1+c_1)^2} \\ &+ \frac{1}{\sqrt{t\,a_2}} {\sum_{n_1\in\mathbf{Z}}}' {\sum_{\tilde{n}_2\in\mathbf{Z}}}' e^{-\frac{\pi \tilde{n}_2^2}{t\, a_2} - \pi t\, a_1\,(n_1+c_1)^2 + 2\pi i \tilde{n}_2 c_2}\end{aligned} \tag{D.3}$$

The first term has $n_1=0$, the last two have $n_1\neq 0$. Then

$$\mathcal{J}_v = \mathcal{K}_1^{(v)} + \mathcal{K}_2^{(v)} + \mathcal{K}_3^{(v)} \tag{D.4}$$

$\mathcal{K}_i^{(v)}$, are obtained by integrating term-wise (D.3) with appropriate coefficients and extra n_1 dependence, see eqs. (D.5), (D.6), (D.18) below. Their evaluation follows:

Calculation of $\mathcal{K}_1^{(v)}$:

$$\mathcal{K}_1^{(v)} \equiv c_1^v \sum_{n_2\in\mathbf{Z}} \int_0^\infty \frac{dt}{t^{1-\epsilon/2}}\, e^{-\pi\, t\,[a_2(n_2+c_2)^2+a_1 c_1^2]-\pi c t} = -c_1^v \,\ln\Big|2\sin(\pi i \gamma(0))\Big|^2 \tag{D.5}$$

which was computed by first performing a re-summation (E.4) over n_2, and then used the integral representation (E.1) of the Bessel function $K_{\frac{1}{2}}$ its expression (E.2), and (D.2).

Calculation of $\mathcal{K}_2^{(v)}$: Here we distinguish two cases: if $0 < c/a_1 < 1$ one has:

$$\begin{aligned}\mathcal{K}_2^{(v)} &\equiv \frac{1}{\sqrt{a_2}} {\sum_{n_1\in\mathbf{Z}}}' (n_1+c_1)^v \int_0^\infty \frac{dt}{t^{3/2-\epsilon/2}}\, e^{-\pi t\, a_1\,(n_1+c_1)^2 - \pi t\, c} \\ &= \frac{\pi^{\frac{1}{2}-\frac{\epsilon}{2}}}{\sqrt{a_2}}\, \Gamma[-1/2+\epsilon/2] {\sum_{n_1\in\mathbf{Z}}}' (n_1+c_1)^v \Big[c + a_1(n_1+c_1)^2\Big]^{\frac{1}{2}-\frac{\epsilon}{2}} \\ &= \frac{(\pi a_1)^{\frac{1}{2}-\frac{\epsilon}{2}}}{\sqrt{a_2}} \sum_{k\geq 0} \left[\frac{-c}{a_1}\right]^k \frac{\Gamma[k-1/2+\epsilon/2]}{k!} \Big[\zeta[2k-q,1+c_1] + (-1)^v \zeta[2k-q,1-c_1]\Big]\Big|_{q=v+1-\epsilon}\end{aligned} \tag{D.6}$$

where, in the second line above we used the binomial expansion

$$[a(n+c)^2+q]^{-s} = a^{-s}\sum_{k\geq 0}\frac{\Gamma[k+s]}{k\,!\,\Gamma[s]}\left[\frac{-q}{a}\right]^k[(n+c)^2]^{-s-k} \tag{D.7}$$

We employed the Hurwitz Zeta function, $\zeta[z,a] = \sum_{n\geq 0}(a+n)^{-z}$, $a \neq 0,-1,-2,\ldots$ for $\mathrm{Re}(z)>1$. One has $\zeta[z,1] = \zeta[z]$ where $\zeta[z]$ is the Riemann zeta function. Hurwitz zeta-function has one singularity (simple pole) at $z=1$. Therefore, in the last line in (D.6), under the sum, a singularity in Zeta functions is present for those k with $2k-v-1=1$. When present, this singularity is taken care of by the presence of ϵ in the argument of Zeta functions. The presence of such singularity depends on the values of the parameter v. We therefore distinguish below two situations:
(i) $v = -2, 0, 2, 4, 6, 8, \ldots$. when such a singularity is present in the term with $k = v/2+1$.
(ii) when v is different from these values.

In case (ii) the result is already that given by (D.6) where one (is allowed to) sets $\epsilon = 0$ since the series does not develop any singularity and converges rapidly under our initial assumption for the ratio $0 \leq c/a_1 < 1$. For case (i), when a singularity develops, we isolate the corresponding term in the series from the rest, by using

$$\begin{aligned}\zeta[1+\epsilon, 1\pm c_1] &= \frac{1}{\epsilon} - \psi(1\pm c_1) + \mathcal{O}(\epsilon)\\ \Gamma[v+1/2+\epsilon/2] &= \Gamma[v+1/2]\,\Big(1+(\epsilon/2)\,\psi(v+1/2)\Big) + \mathcal{O}(\epsilon^2)\\ x^\epsilon &= 1+\epsilon\ln x + \mathcal{O}(\epsilon)\end{aligned} \tag{D.8}$$

with $\psi(z) = (d/dz)\ln\Gamma[z]$ the Digamma function. In the remaining terms in the series we are allowed to take $\epsilon \to 0$. We find that for $v = -2, 0, 2, 4, 6, \ldots$

$$\begin{aligned}\mathcal{K}_2^{(v)} = &\sqrt{\pi}u\sum_{k\geq 0}\frac{\Gamma[k-1/2]}{k!}\left[\frac{-c}{a_1}\right]^k\Big[\zeta[2k-v-1,1+c_1]+\zeta[2k-v-1,1-c_1]\Big]\Big|_{k\neq v/2+1}\\ &-\sqrt{\pi}\,u\,\frac{\Gamma[v/2+1/2]}{(v/2+1)!}\left[\frac{-c}{a_1}\right]^{v/2+1}\left[\frac{-2}{\epsilon}+\ln\Big[\pi a_1 e^{-\psi(v/2+1/2)+\psi(c_1)+\psi(-c_1)}\Big]\right], \quad u\equiv\sqrt{a_1/a_2}\end{aligned} \tag{D.9}$$

where the series converges quickly if $|c/a_1| < 1$, which justifies our (stronger) initial assumption $0 \leq c/a_1 < 1$. This concludes the discussion for case (i).

Replacing now $v = 0, 1, 2$ in the above result, one obtains the appropriate expressions for $\mathcal{K}^{(0)}$, $\mathcal{K}^{(1)}$ and $\mathcal{K}^{(2)}$, that we need for our purposes. One has

$$\begin{aligned}\mathcal{K}_2^{(0)} = &\frac{\pi\, c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon}+\ln\Big[4\pi\, a_1\, e^{\gamma_E+\psi(c_1)+\psi(-c_1)}\Big]\right]+2\pi\, u\Big(\frac{1}{6}+c_1^2\Big)\\ &+\sqrt{\pi}\,u\sum_{p\geq 1}\frac{\Gamma[p+1/2]}{(p+1)!}\left[\frac{-c}{a_1}\right]^{p+1}\Big(\zeta[2p+1,1+c_1]+\zeta[2p+1,1-c_1]\Big),\ u\equiv\left[\frac{a_1}{a_2}\right]^{\frac{1}{2}}\end{aligned} \tag{D.10}$$

and

$$\mathcal{K}_2^{(1)} = \sqrt{\pi}\, u \sum_{p\geq 0} \frac{\Gamma[p+3/2]}{(p+2)!} \left[\frac{-c}{a_1}\right]^{p+2} \Big(\zeta[2p+2, 1+c_1] - \zeta[2p+2, 1-c_1]\Big)$$

$$+ 2\pi\, u\, c_1 \Big[\frac{1}{3}\,(1+2c_1^2) + \frac{c}{a_1}\Big], \qquad u \equiv \sqrt{a_1/a_2} \tag{D.11}$$

Finally

$$\mathcal{K}_2^{(2)} = \pi u \left[\frac{-1}{30} + c_1^2 + c_1^4\right] + \frac{\pi\, c}{\sqrt{a_1 a_2}} \Big[\frac{1}{6} + c_1^2\Big] - \frac{\pi\, c^2}{4 a_1 \sqrt{a_1 a_2}} \Big[\frac{-2}{\epsilon} + \ln\Big[4\pi a_1\, e^{\gamma_E - 2 + \psi(c_1) + \psi(-c_1)}\Big]\Big]$$

$$+ \sqrt{\pi}\, u \sum_{p>0} \frac{\Gamma[p+3/2]}{(p+2)!} \left[\frac{-c}{a_1}\right]^{p+2} \Big(\zeta[2p+1, 1+c_1] + \zeta[2p+1, 1-c_1]\Big), \tag{D.12}$$

In the remaining case $1 \leq c/a_1$ we examine separately the cases $v = 0, 1, 2$. One shows:

$$\mathcal{K}_2^{(0)} \equiv \sum_{n_1 \in \mathbf{Z}}{}' \frac{1}{\sqrt{a_2}} \int_0^\infty \frac{dt}{t^{3/2-\epsilon/2}}\, e^{-\pi t a_1 (n_1+c_1)^2 - \pi\, t\, c} \tag{D.13}$$

$$= \frac{\pi c}{\sqrt{a_1 a_2}} \Big[\frac{-2}{\epsilon} + \ln(\pi\, c\, e^{\gamma_E - 1})\Big] + 4\left[\frac{c}{a_2}\right]^{\frac{1}{2}} \sum_{\tilde{n}_1 > 0} \frac{\cos(2\pi \tilde{n}_1 c_1)}{\tilde{n}_1} K_1\Big(2\pi \tilde{n}_1 \sqrt{\frac{c}{a_1}}\Big) + \frac{2\pi}{\sqrt{a_2}} (c + a_1 c_1^2)^{\frac{1}{2}}$$

This expression was obtained by firstly adding and subtracting a zero mode, which enabled us to then re-sum (see (E.4)) the series over $n_1 \in \mathbf{Z}$. We then used the integral representation of the modified Bessel functions K_1 (E.1). The pole present is that of the initial "missing" zero mode. The presence of the Bessel function $K_1[z]$ which is exponentially suppressed (E.2) ensures that the result above converges rapidly in this case too.

One also has, for $v = 1$ (again $1 \leq c/a_1$):

$$\mathcal{K}_2^{(1)} \equiv \frac{1}{\sqrt{a}_1} \sum_{n_1 \in \mathbf{Z}}{}' (n_1 + c_1) \int_0^\infty \frac{dt}{t^{3/2-\epsilon/2}}\, e^{-\pi t a_1 (n_1+c_1)^2 - \pi t c} \tag{D.14}$$

$$= -\frac{1}{2 a_1 \pi} \frac{1}{\sqrt{a}_2} \frac{\partial}{\partial c_1} \sum_{n_1 \in \mathbf{Z}}{}' \int_0^\infty \frac{dt}{t^{5/2-\epsilon/2}}\, e^{-\pi t a_1 (n_1+c_1)^2 - \pi\, t\, c}$$

$$= -\frac{1}{2 a_1 \pi} \frac{1}{\sqrt{a}_2} \frac{\partial}{\partial c_1} \Big\{ -\frac{\pi^2 c^2}{2\sqrt{a}_1} \Big[\frac{-2}{\epsilon} + \ln\big(\pi\, c\, e^{\gamma_E - 3/2}\big)\Big]$$

$$+ 4\, c\, \sqrt{a_1} \sum_{\tilde{n}_1 > 0} \frac{\cos(2\pi \tilde{n}_1 c_1)}{\tilde{n}_1^2} K_2(s_{\tilde{n}_1}) - \frac{4\pi^2}{3} (c + a_1 c_1^2)^{\frac{3}{2}} \Big\}$$

$$= \frac{4c}{\sqrt{a_1 a_2}} \sum_{\tilde{n}_1 > 0} \frac{\sin(2\pi \tilde{n}_1 c_1)}{\tilde{n}_1} K_2(s_{\tilde{n}_1}) + \frac{2\pi c_1}{\sqrt{a_2}} (c + a_1 c_1)^{\frac{1}{2}}, \quad s_{\tilde{n}_1} \equiv 2\pi \tilde{n}_1 \sqrt{c/a_1} \tag{D.15}$$

where the series converges rapidly, due to exponential suppression of the Bessel function K_2. To evaluate the integral over t with denominator $t^{5/2-\epsilon/2}$ one uses steps identical to those for $\mathcal{K}_2^{(0)}$ with the only difference that we encountered an integral representation of K_2 rather than K_1.

Finally, for the remaining case $v=2$ $(1\leq c/a_1)$:

$$
\begin{aligned}
\mathcal{K}_2^{(2)} &\equiv \frac{1}{\sqrt{a_1}} \sideset{}{'}\sum_{n_1\in \mathbf{Z}} (n_1+c_1)^2 \int_0^\infty \frac{dt}{t^{3/2-\epsilon/2}}\, e^{-\pi t a_1 (n_1+c_1)^2-\pi\, t\, c} \qquad\qquad (D.16)\\
&= -\frac{1}{\pi}\frac{1}{\sqrt{a_2}}\frac{\partial}{\partial a_1} \sideset{}{'}\sum_{n_1\in \mathbf{Z}} \int_0^\infty \frac{dt}{t^{5/2-\epsilon/2}}\, e^{-\pi t a_1 (n_1+c_1)^2-\pi\, t\, c} = -\frac{1}{\pi}\frac{1}{\sqrt{a_2}}\frac{\partial}{\partial a_1}\bigg\{\frac{-4\pi^2}{3}(c+a_1c_1^2)^{\frac{3}{2}}\\
&\quad - \frac{\pi^2 c^2}{2\sqrt{a_1}}\Big[\frac{-2}{\epsilon}+\ln\big(\pi\, c\, e^{\gamma_E-\frac{3}{2}}\big)\Big] - \frac{4\pi^2}{3}(c+a_1c_1^2)^{\frac{3}{2}} + 4c\sqrt{a_1}\sum_{\tilde n_1>0}\frac{\cos(2\pi\tilde n_1 c_1)}{\tilde n_1^2}K_2(s_{\tilde n_1})\bigg\}\\
&= \frac{-\pi\, c^2}{4a_1\sqrt{a_1a_2}}\Big[\frac{-2}{\epsilon}+\ln\Big(\pi\, c\, e^{\gamma_E-\frac{3}{2}}\Big)\Big] - \frac{2c}{\pi\sqrt{a_1a_2}}\sum_{\tilde n_1>0}\frac{\cos(2\pi\tilde n_1c_1)}{\tilde n_1^2}\Big[3K_2(s_{\tilde n_1})+s_{\tilde n_1}K_1(s_{\tilde n_1})\Big]\\
&\quad + \frac{2\pi c_1^2}{\sqrt{a_2}}(c+a_1c_1^2)^{\frac{1}{2}}, \qquad\qquad s_{\tilde n_1}\equiv 2\pi\tilde n_1\sqrt{c/a_1}; \qquad c/a_1\geq 1. \qquad\qquad (D.17)
\end{aligned}
$$

with intermediate steps similar to those for $\mathcal{K}_2^{(1)}$.

Calculation of $\mathcal{K}_3^{(v)}$: Finally, we evaluate the remaining:

$$
\begin{aligned}
\mathcal{K}_3^{(v)} &\equiv \frac{1}{\sqrt{a_2}} \sideset{}{'}\sum_{n_1\in \mathbf{Z}} \sideset{}{'}\sum_{\tilde n_2\in \mathbf{Z}} (n_1+c_1)^v \int_0^\infty \frac{dt}{t^{3/2-\epsilon/2}}\, e^{-\frac{\pi\tilde n_2^2}{t\, a_2}-\pi t\, a_1\, (n_1+c_1)^2+2\pi i\tilde n_2\, c_2-\pi\, t\, c} \qquad (D.18)\\
&= \frac{1}{\sqrt{a_2}} \sideset{}{'}\sum_{n_1\in \mathbf{Z}} \sum_{\tilde n_2>0} (n_1+c_1)^v \frac{1}{\tilde n_2}\, e^{-2\pi\tilde n_2\,\gamma(n_1)} + c.c.\\
&= -\sideset{}{'}\sum_{n_1\in \mathbf{Z}} (n_1+c_1)^v \ln\Big|1-e^{-2\pi\gamma(n_1)}\Big|^2\\
&= -\sum_{n_1\in \mathbf{Z}} (n_1+c_1)^v \ln\Big|1-e^{-2\pi\gamma(n_1)}\Big|^2 - \frac{2\pi c_1^v}{\sqrt{a_2}}(c+a_1c_1^2)^{\frac{1}{2}} + c_1^v\ln\Big|2\sin(\pi i\gamma(0))\Big|^2 \qquad (D.19)
\end{aligned}
$$

using the notations in eq. (D.2). In the last line we re-wrote the result in a form which makes explicit the cancellations which occur in the sum of $\mathcal{J}_v = \mathcal{K}_1^{(v)}+\mathcal{K}_2^{(v)}+\mathcal{K}_3^{(v)}$.

The steps in the calculation of $\mathcal{K}_3^{(v)}$ are similar to those so far: we used the integral representation of the Bessel function $K_{1/2}$ eq. (E.1), then its explicit expression (E.2) and then the series expansion of the logarithm. The result for $\mathcal{K}_3^{(v)}$ is valid for real v, not only for our cases of interest $v=0,1,2$, regardless of the value c/a_1 (larger/smaller than 1).

We can now add the intermediate eqs to obtain $\mathcal{J}_{0,1,2}$ using eq. (D.4). $\mathcal{J}_0$ quoted below in (D.20) and (D.21) is found from eqs. (D.5), (D.10), (D.13), (D.19). Further, $\mathcal{J}_1$ quoted in (D.23) and (D.24) is found using eqs. (D.5), (D.11),(D.15), (D.19). Finally $\mathcal{J}_2$ quoted in (D.25) and (D.26) is obtained by using (D.5), (D.12), (D.17), (D.19). In conclusion we have the following:

Results: If $0 \leq c/a_1 < 1$ and with notations (D.2), $\gamma(n_1) \equiv \sqrt{z(n_1)}/\sqrt{a_2} - i\,c_2$; and $z(n_1) \equiv c + a_1(n_1+c_1)^2$, $u \equiv \sqrt{a_1/a_2}$, $s_{\tilde{n}_1} \equiv 2\pi\tilde{n}_1\sqrt{c/a_1}$, $\gamma_E = 0.577216\ldots$ we obtain (in the text $a_1 = 1/R_5^2$, $a_2 = 1/R_6^2$

$$\mathcal{J}_0[c;c_1,c_2] = \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon} + \ln\left[4\pi\,a_1\,e^{\gamma_E+\psi(c_1)+\psi(-c_1)}\right]\right] + 2\pi\,u\left[\frac{1}{6} + c_1^2 - \left(c/a_1 + c_1^2\right)^{\frac{1}{2}}\right]$$
$$-\sum_{n_1\in\mathbf{Z}} \ln\left|1 - e^{-2\pi\,\gamma(n_1)}\right|^2 + \sqrt{\pi}\,u\sum_{p\geq 1}\frac{\Gamma[p+1/2]}{(p+1)!}\left[\frac{-c}{a_1}\right]^{p+1}\Big(\zeta[2p+1,1+c_1] + \zeta[2p+1,1-c_1]\Big) \quad \text{(D.20)}$$

while if we have $c/a_1 > 1$, then

$$\mathcal{J}_0[c;c_1,c_2] = \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon} + \ln\left[\pi\,c\,e^{\gamma_E - 1}\right]\right] - \sum_{n_1\in\mathbf{Z}}\ln\left|1 - e^{-2\pi\,\gamma(n_1)}\right|^2$$
$$+ \frac{4\sqrt{c}}{\sqrt{a_2}}\sum_{\tilde{n}_1>0}\frac{\cos(2\pi\tilde{n}_1\,c_1)}{\tilde{n}_1}\,K_1(s_{\tilde{n}_1}) \quad \text{(D.21)}$$

The pole structure is the same for both cases; if $c/a_1 > 1$ and except the first square bracket, no power-like terms in c are present (the last one being suppressed due to K_1).

Finally, we quote here a limiting case for the behaviour of the function $\mathcal{J}_0$

$$\mathcal{J}_0[c \ll 1;0,0] = \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon} + \ln\left[4\pi e^{-\gamma_E}a_1\left|\eta(i\sqrt{a_1/a_2})\right|^4\right]\right]$$
$$- \ln\left[4\pi^2\,|\eta(i\sqrt{a_1/a_2})|^4\,a_2^{-1}\right] - \ln c \quad \text{(D.22)}$$

and this was used in the text in eq. (3.57).

Further, if $0 \leq c/a_1 < 1$

$$\mathcal{J}_1[c,c_1,c_2] = 2\pi c_1\,u\left[\frac{c}{a_1} - (c/a_1 + c_1^2)^{\frac{1}{2}} + \frac{1}{3}(1+2c_1^2)\right] - \sum_{n_1\in\mathbf{Z}}(n_1+c_1)\,\ln\left|1 - e^{-2\pi\gamma_{n_1}}\right|^2$$
$$+ \sqrt{\pi}\,u\sum_{p\geq 0}\frac{\Gamma(p+3/2)}{(p+2)!}\left[\frac{-c}{a_1}\right]^{p+2}\Big(\zeta[2p+2,1+c_1] - \zeta[2p+2,1-c_1]\Big) \quad \text{(D.23)}$$

while if $c/a_1 > 1$, then

$$\mathcal{J}_1[c,c_1,c_2] = -\sum_{n_1\in\mathbf{Z}}(n_1+c_1)\,\ln\left|1 - e^{-2\pi\gamma(n_1)}\right|^2 + \frac{4\,c}{\sqrt{a_1 a_2}}\sum_{\tilde{n}_1>0}\frac{\sin(2\pi\tilde{n}_1 c_1)}{\tilde{n}_1}\,K_2(s_{\tilde{n}_1}) \quad \text{(D.24)}$$

where $s_{\tilde{n}_1} \equiv 2\pi\tilde{n}_1\sqrt{c/a_1}$. Note that $\mathcal{J}_1$ has no poles in ϵ, unlike the case of $\mathcal{J}_{0,2}$. K_1 is exponentially suppressed at large argument.

Finally, if $0 \leq c/a_1 < 1$

$$\begin{aligned}\mathcal{J}_2[c,c_1,c_2] &= -\frac{\pi c^2}{4\,a_1\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon}+\ln\left[4\pi\,a_1\,e^{\gamma_E+\psi(c_1)+\psi(-c_1)-2}\right]\right]\\ &\quad -\pi\,u\left[\frac{1}{30}-\frac{c}{6a_1}-c_1^2\Big(1-(c/a_1+c_1^2)^{\frac{1}{2}}\Big)^2\right]-\sum_{n_1\in\mathbf{Z}}(n_1+c_1)^2\ln\left|1-e^{-2\pi\gamma(n_1)}\right|^2\\ &\quad +\sqrt{\pi}\,u\sum_{p\geq 1}\frac{\Gamma[p+3/2]}{(p+2)!}\left[\frac{-c}{a_1}\right]^{p+2}\Big(\zeta[2p+1,1+c_1]+\zeta[2p+1,1-c_1]\Big). \end{aligned} \tag{D.25}$$

while if $c/a_1 > 1$ then:

$$\begin{aligned}\mathcal{J}_2[c,c_1,c_2] &= -\frac{\pi c^2}{4a_1\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon}+\ln\left[\pi\,c\,e^{\gamma_E-3/2}\right]\right]-\sum_{n_1\in\mathbf{Z}}(n_1+c_1)^2\ln\left|1-e^{2\pi\gamma(n_1)}\right|^2\\ &\quad -\frac{2\,c}{\pi\sqrt{a_1 a_2}}\sum_{\tilde{n}_1>0}\frac{\cos(2\pi\,\tilde{n}_1 c_1)}{\tilde{n}_1^2}\left[3K_2(s_{\tilde{n}_1})+s_{\tilde{n}_1}K_1(s_{\tilde{n}_1})\right], \end{aligned} \tag{D.26}$$

where $s_{\tilde{n}_1} \equiv 2\pi\tilde{n}_1\sqrt{c/a_1}$.

The series with zeta functions converge under the assumption $0 \leq c/a_1 < 1$. The presence of Bessel functions $K_{1,2}$ (see (E.2)) which are exponentially suppressed with respect to their argument (larger than unity) ensures a rapid convergence of the corresponding series. Similar expressions exist for $\mathcal{I}_v = \mathcal{J}_v|_{c_1\leftrightarrow c_2; a_1\leftrightarrow a_2}$; and are obtained from those above with replacements $a_1 \leftrightarrow a_2$, $c_1 \leftrightarrow c_2$.

E. Definitions of special functions

The modified Bessel functions $K_n(z)$ used above have the integral representation/definition:

$$\int_0^\infty dx\,x^{\nu-1}e^{-bx^p-ax^{-p}} = \frac{2}{p}\left[\frac{a}{b}\right]^{\frac{\nu}{2p}}K_{\frac{\nu}{p}}(2\sqrt{a\,b}), \quad \mathrm{Re}(b),\,\mathrm{Re}(a)>0 \tag{E.1}$$

with

$$\begin{aligned} K_1[x] &= e^{-x}\sqrt{\frac{\pi}{2x}}\left[1+\frac{3}{8x}-\frac{15}{128x^2}+\mathcal{O}(1/x^3)\right]\\ K_2[x] &= e^{-x}\sqrt{\frac{\pi}{2x}}\left[1+\frac{15}{8x}+\frac{105}{128}\frac{1}{x^2}+\mathcal{O}(1/x^3)\right]\\ K_{\frac{1}{2}}[x] &= e^{-x}\sqrt{\frac{\pi}{2x}}\\ K_{\frac{3}{2}}[x] &= e^{-x}\sqrt{\frac{\pi}{2x}}\left[1+\frac{1}{x}\right] \end{aligned} \tag{E.2}$$

The definition of the poly-logarithm function used above

$$\mathrm{Li}_\sigma(x) = \sum_{x\geq 1} \frac{x^n}{n^\sigma} \tag{E.3}$$

The one-dimensional Poisson re-summation used in the appendix:

$$\sum_{n\in Z} e^{-\pi A(n+\sigma)^2} = \frac{1}{\sqrt{A}} \sum_{\tilde{n}\in Z} e^{-\pi A^{-1}\tilde{n}^2 + 2i\pi\tilde{n}\sigma} \tag{E.4}$$

The Hurwitz Zeta function used in this paper is defined as

$$\zeta[z,a] = \sum_{n\geq 0} (a+n)^{-z} \tag{E.5}$$

where $a \neq 0, -1, -2, \ldots$ for $\mathrm{Re}(z) > 1$. One has $\zeta[z,1] = \zeta[z]$ where $\zeta[z]$ is the Riemann zeta function. Hurwitz zeta-function has one singularity (simple pole) at $z = 1$.

We also used the Dedekind function

$$\begin{aligned} \eta(\tau) &\equiv e^{\pi i\tau/12} \prod_{n\geq 1}(1 - e^{2i\pi\tau\, n}), \\ \eta(-1/\tau) &= \sqrt{-i\,\tau}\,\eta(\tau), \qquad \eta(\tau+1) = e^{i\pi/12}\eta(\tau). \end{aligned} \tag{E.6}$$

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